Diff Geo Lecture 2

Recall

- Given M, we can construct the tangent bundle TM, which comes equipped with a map $\pi: TM \to M$ where $\forall p \in M$, $\pi^{-1}(p) = T_pM$. In addition, given coordinates x^i on M, we get coordinates on TM by $(p,v) \mapsto (x^1(p) \dots x^n(p), v^1 \dots v^n)$

  where $v = v^i \partial_i$

A local section of TM is a map $\sigma: U \to TM$ s.t.

$\pi \circ \sigma = id_U$

- Essentially assigning to each $p \in U$ a vector $v \in T_pM$

Def: A vector field on M is a global section $X: M \to TM$. We often denote $X(p) = X_p \in T_pM$, and the set of all vector fields as $\mathfrak{X}(M)$.

We already know local vector fields - given coords x^i, we have the coordinate vector fields $\frac{\partial}{\partial x^i}$ where

$$\left(\frac{\partial}{\partial x^i}\right)\Big|_p = \frac{\partial}{\partial x^i}\Big|_p$$

We know that for all $p \in U$, the $\frac{\partial}{\partial x^i}\Big|_p$ form a basis $\Rightarrow$ any smooth vector field X can be written locally as $X^i \frac{\partial}{\partial x^i}$ for smooth functions $X^i : U \to \mathbb{R}$ called the <u>component functions</u>

$\mathfrak{X}(M)$ has more structure - it is a vector space. (in fact, a $C^\infty(M)$-module)

given $X \in \mathfrak{X}(M)$, fX is the vector field

$$(fX)_p = f(p) X_p.$$

Recall that tangent vectors act on functions - they are derivations $C^\infty(M) \to \mathbb{R}$.

By acting pointwise, this means that vector <u>fields</u> act on $C^\infty(M)$ as well.

Define $Xf \in C^\infty(M)$ by

$$(Xf)(p) = X_p f \in \mathbb{R}$$

In this way, vector fields are <u>derivations</u>

i.e. $X(fg) = fXg + gXf$

Note that where a function appears matters

fX is a vector field, Xf is a function

As with our example w/ vectors/derivations on $\mathbb{R}^n$, there is a bijection $\{\text{Vector fields}\} \longleftrightarrow \{\text{derivations on } C^\infty(M)\}$

$X \longmapsto D_X$ where $D_X f = Xf$

Smooth maps F and tangent vectors interact via the derivative dF_p. The same applies to vector fields.

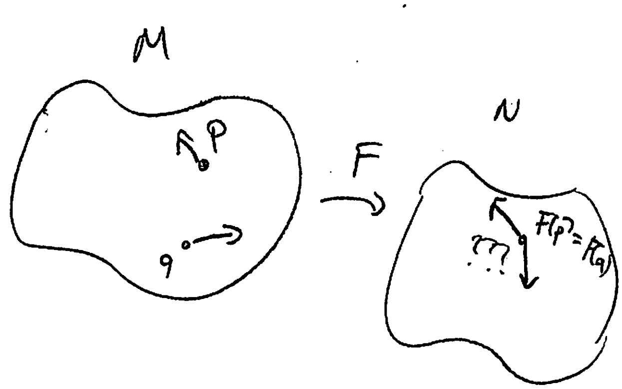
Given $X \in \mathfrak{X}(M)$, $F: M \to N$, we want to define a vector field Y on N.

The obvious thing - $Y_p = dF_{F^{-1}(p)}(X_{F^{-1}(p)})$

Two problems

1) $F^{-1}(p) = \emptyset$

2) $|F^{-1}(p)| > 1$



<u>Def:</u> Given $F: M \to N$, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are <u>F-related</u> if $\forall p \in M$,

$$dF_p(X_p) = Y_{F(p)}$$

In the case that F is a diffeomorphism, there is a unique such vector field, denoted $(F_* X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)})$ called the <u>pushforward</u>

Vector fields have a nice operation, called the <u>Lie Bracket</u> $[-,-]: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

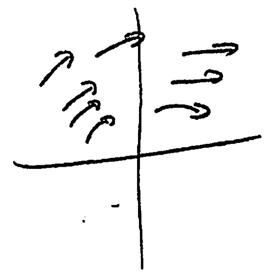
Defined by $[X,Y]f = XYf - YXf$

in coordinates, $[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}$

Brackets respect F-relatedness, in particular, it commutes with pushforward - $F_*[X,Y] = [F_* X, F_* Y]$

Flows

In $\mathbb{R}^n$, vector fields are like currents infinitesimally describe a motion



Taking infinitesimal steps in the direction of the vector field

Def: For a vector field $V \in \mathfrak{X}(M)$, a curve $\gamma: I \to M$ is an integral curve of V if

$$\dot{\gamma}(t) = V_{\gamma(t)}$$

V is said to be the velocity vector field for γ

In coordinates, $\dot{\gamma}(t) = V_{\gamma}(t)$ becomes

$$\dot{\gamma}^i(t)\frac{\partial}{\partial x^i} = V^i(\gamma(t))\frac{\partial}{\partial x^i}$$

Which is a system of ODE. We know how to solve these in $\mathbb{R}^n$ (Existence + uniqueness thm)

and solving them is a local problem

$\Rightarrow$ We can do it on mflds via charts

Thm: Given a smooth vector field V, for all $p \in M$, there exists an interval $(-\varepsilon, \varepsilon)$ and a smooth integral curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$

So given a vector field V, we can (usually) "put" these integral curves together into a diffeomorphism.

Def: A global flow on M is a smooth map

$$\theta : \mathbb{R} \times M \to M \quad \text{s.t.}$$

1) $\theta(t, \theta(s,p)) = \theta(t+s, p)$

$\theta(0,p) = p.$

To end, Lie derivatives? Frobenius' theorem??

i.e. a group action $\mathbb{R}^+ \curvearrowright M$.

A flow determines maps in two ways

$$\begin{aligned} \theta_t : M &\to M \\ p &\mapsto \theta(t,p) \end{aligned}$$

$$\begin{aligned} \theta^{(p)} : \mathbb{R} &\to M \\ t &\mapsto \theta(t,p) \end{aligned}$$

$\theta^{(p)}$ gives a vector field by differentiating $V_p = \dot{\theta}^{(p)}(0)$