DIFFERENTIAL GEOMETRY LECTURE 2

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1. Vector Fields

Recall that given a smooth n dimensional manifold M, we get the ***tangent bundle*** TM, which is a $2n$ dimensional smooth manifold, equipped with a map $\pi : TM \to M$, where the fiber $\pi^{-1}(p)$ over a point $p \in M$ is the tangent space T_pM. Also recall that a ***section*** of a π is a map $\sigma : U \to TM$ such that $\pi \circ \sigma = \mathrm{id}_U$. If $U = M$, we call σ a global section.

Definition 1.1. A ***vector field*** on a smooth manifold is a global section $X : M \to TM$. The set of vector fields on M is denoted $\mathfrak{X}(M)$.

If you unwrap the definitions, we see that a section is exactly the data we want – for every point $p \in M$, we are assigning to it a vector in T_pM. We will often denote the vector $X(p)$ at p by X_p.

If we fix a chart U with coordinates x^i, we get the ***coordinate vector fields*** $\partial/\partial x^i$, where

$$\left(\frac{\partial}{\partial x^i}\right)_p = \left.\frac{\partial}{\partial x^i}\right|_p$$

Then given an arbitrary smooth vector field X, we have that X_p is a linear combination of the coordinate vector fields. Smoothness of X then tells us that we can write X locally as

$$X = X^i \frac{\partial}{\partial x^i}$$

for smooth functions $X^i : U \to \mathbb{R}$. We also note that $\mathfrak{X}(M)$ admits more structure than that of a set.

Proposition 1.2.

(1) *Let $X, Y \in \mathfrak{X}(M)$. Then $fX + gY$ defined a smooth vector field, for $f, g \in C^\infty(M)$.*
(2) *$\mathfrak{X}(M)$ is a $C^\infty(M)$ module.*

In more sheafy language, this tells us that the sheaf of vector fields on M is a sheaf of modules over the sheaf of smooth functions. In fact, we can say even more about the algebraic structure of vector fields. Given a chart U with coordinates x^i, we know we can write any vector field as $X^i\partial_i$. This tells us that the ∂_i form a basis for the local vector fields $\mathfrak{X}(U)$ as a $C^\infty(U)$ module, i.e. locally, the vector fields form a free module over $C^\infty(U)$. Note that this might not hold globally though.

Definition 1.3. A ***local frame*** for M is an collection of smooth vector fields E_i defined on an open set $U \subset M$ such that for each $p \in U$, we have that the $E_i|_p$ form a basis for T_pM. If $U = M$, we say that the E_i form a ***global frame***.

We've already seen a local frame, the coordinate vector fields ∂_i.

Recall that a vector $v \in T_pM$ acts on functions $f \in C^\infty(M)$ – it takes as an input a smooth function, and produces a real number. We see then that vector fields act on smooth functions as well, where we define the action pointwise to produce a new function. Explicity, given $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we have

$$(Xf)(p) = X_pf$$

In this way, we see that a vector field X determines a linear map $C^\infty(M) \to C^\infty(M)$. In fact, it defines a ***derivation***, i.e.

$$X(fg) = fXg + gXf$$

since each vector X_p is a derivation at p.

Proposition 1.4. *Vector fields on M are in bijection with derivations* $D : C^\infty(M) \to C^\infty(M)$*, where the mapping is given by* $X \mapsto D_X$ *where* $D_X f = Xf$*.*

Given a vector field $X \in \mathfrak{X}(M)$ and a smooth map $F : M \to N$, we can apply the differential dF_p pointwise to X, but the result may not be well defined. For example, if F is not injective, there will exists at least two points p, q such that $F(p) = F(q) = y$. Then if we want to use F and X to define a vector field on Y, we have a conundrum – what vector should we assign to y? Should it be $dF_p(X_p)$ or $dF_q(X_q)$? Therefore in order for X to push forward to a vector field on Y, we must impose the condition on each fiber $F^{-1}(y)$ that for all $p \in F^{-1}(y)$, we have $dF_p(X_p)$ is the same.

Definition 1.5. Given smooth manifolds M and N, a smooth map $F : M \to N$, and vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. We say that X and Y are ***F-related*** if for all $q \in N$ and all $p \in F^{-1}(q)$, we have $dF_p(X_p) = Y_q$.

Given an arbitrary vector field X and a smooth map $F : M \to N$, it's not true in general that an F-related vector field exists in $\mathfrak{X}(N)$, however, in the case that F is a diffeomorphism, a unique F-related vector field exists, called the pushforward F_*X. In order for a vector field to be F-related to X, we see that it must be define by

$$(F_*X)_p = dF_{F^{-1}(p)}(X_{F^{-1}(p)})$$

which is well defined since F is invertible. The set of vector fields $\mathfrak{X}(M)$ already carries a great deal of rich algebraic structure. It is a vector space, a $C^\infty(M)$ module, and the space of derivations on $C^\infty(M)$. It also has another kind of algebraic structure, that of a *Lie algebra*.

Definition 1.6. The ***Lie bracket*** of vector fields is the bilinear map

$$[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

where given $X, Y \in \mathfrak{X}(M)$, the vector field $[X, Y]$ is defined by

$$[X, Y]f = XYf - YXf$$

There's a little to unpack with the defintion, namely, what do the terms XYf and YXf actually mean? Recall that vector fields eat functions and produce new ones, so Yf is some smooth function. Therefore, we can feed this function into X to get another function. Doing this in the opposite order gives us YXf and their difference is the action of the Lie bracket of X and Y. If we have local coordinates x^i, then the vector fields X and Y have the coordinate formulas

$$X = X^i \frac{\partial}{\partial x^i} \qquad Y = Y^i \frac{\partial}{\partial x^i}$$

Then the Lie bracket has the coordinate representation

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} = \left(XY^j - YX^j \right) \frac{\partial}{\partial x^j}$$

One thing to note is that the coordinate vector fields satisfy $[\partial_i, \partial_j] = 0$, since all the component functions are constant. In some sense, this is the defining feature of the coordinate vector fields.

Theorem 1.7. *The Lie bracket is natural in the following sense. Let* $F : M \to N$ *be a smooth map,* $X_1, X_2 \in \mathfrak{X}(M)$ *and* $Y_1, Y_2 \in mathfrakX(N)$ *such that* Y_1 *is F-related to* X_1 *and* Y_2 *is F-related to* X_2*. Then* $[Y_1, Y_2]$ *is F-related to* $[X_1, F_2]$*. In the case that F is a diffeomorphism, this says that* $[\cdot, \cdot]$ *commutes with pushforward, i.e.*

$$F_*[X, Y] = [F_*X, F_*Y]$$

2. Flows

In $\mathbb{R}^n$, when we have a vector field X, we can "integrate" the to produce curves. The intution here is that a vector field gives us infinitesimal directions of how to move (like a current in a stream). At a point p, the vector vector X_p tells us which direction to move. After taking a small step, we arrive at a new point q, and then look at X_q for the new direction to step in. This intuition shows us that integrating vector fields to curves is a matter of differential equations. We want a function f such that when we differentiate it, we recover the vector field X. One important thing to note here is that solving differential equations is a *local* condition. To integrate a vector field X near p, we don't need to know the behavior of X outside of some small neighborhood of p. Therefore, translating this to manifolds should go without a hitch. To find integral curves of $X \in \mathfrak{X}(M)$, we can pull the picture back to Eucldiean space with charts, and then the solutions back up the manifold after using our knowledge of differential equations in $\mathbb{R}^n$.

Definition 2.1. Given a vector field $V \in \mathfrak{X}(M)$, a curve $\gamma : I \to M$ is an ***integral curve*** of V if for all $t \in I$, we have

$$\dot{\gamma}(t) = V_{\gamma(t)}$$

We often call V the ***velocity vector field*** of γ.

If we write the vector field V and the curve γ in coordinates x^i, then $\dot{\gamma}(t) = V_{\gamma(t)}$ if and only if

$$\dot{\gamma}^i \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} = V^i(\gamma(t)) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}$$

Therefore, finding such a γ in these coordinates is equivalent to solving the system of differential equations $\dot{\gamma}^i(t) = V^i(\gamma^1(t), \dots, \gamma^n(t))$. From the theory of differential equations, we know this is possible in a small neighborhood of p.

Theorem 2.2 (***Existence and Uniqueness Theorem for ODEs***)**.** *Let $U \subset \mathbb{R}^n$ be open, and let $V : U \to \mathbb{R}^n$ be a smooth function. This determines an initial value problem $\dot{\gamma}(t) = V^i(\gamma^1(t), \dots, \gamma^n(t))$ with intial condition $\gamma(t_0) = c$ for fixed constants $t_0 \in \mathbb{R}$ and $c \in U$. Then for any $t_0 \in \mathbb{R}$ and $x_0 \in U$, there exists some interval J and a smaller neighborhood $V \subset U$ of x_0 such that there exists a continuously differentiable solution $\gamma : J_0 \to U$ for any intial condition $\gamma(t_0) = c$ for $c \in V$. In addition, this solution is unique, and the map $\theta : J \times V \to U$ defined by $\theta(t, x) = \gamma(t)$ where γ is the unique solution with initial condition $\gamma(t_0) = x$ is smooth.*

Corollary 2.3. *Given smooth vector field $V \in \mathfrak{X}(M)$, for each point $p \in M$ there exists some $\varepsilon > 0$ and a smoooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at p.*

As you can imagine, F-related vector fields generate "F-related" flows.

Proposition 2.4. *Given a smooth map $F : M \to N$, vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if an only if F maps integral curves of X to integral curves of Y, i.e. given an integral curve γ for X, $F \circ \gamma$ is an integral curve of Y.*

Definition 2.5. A ***global flow*** on a manifold M is a smooth map $\theta : \mathbb{R} \times M \to M$ such that

(1) $\theta(t, \theta(s, p)) = \theta(t + s, p)$

(2) $\theta(0, p) = p$

in other words, it is a group action of the additive group $\mathbb{R}$ on M.

The intuition here is that given a point p, we can take the integral curve γ starting at p and flow for time t. Following this by following with the integral curve at $\gamma(t)$ for time s, this shoulds be the same as flowing along γ for time $t + s$.

Given a global flow θ, we have two different times to obtain a map. If we fix a point $p \in M$, we obtain the orbit map $\theta^{(p)} : \mathbb{R} \to M$ given by $t \mapsto \theta(t, p)$, and given a number $t \in \mathbb{R}$, we can define the map θ_t by $\theta_t(p) = \theta(t, p)$. Using this, we can define a vector field V by

$$V_p = \dot{\theta}^{(p)}(0)$$

You would like to say that vector fields and flows are inverses to each other – Given a flow, we can define a vector field by differentiating integral curves, and given a vector field, we can integrate it into a flow. Unfortunately, this isn't exactly true. When we integrate a vector field it's not true that the flow is defined for all time. For a simple example, the vector field ∂_x on$\mathbb{R}^2 - \{0\}$ is not complete, if we start flowing at a point p on the negative x-axis, then we cannot flow past $(0, 0)$. That being said, we won't concern ourselves too much about this, and assume that flows are global.

The naturality of integral curves also tells us that flows are natural in the follwing sense

Theorem 2.6 (***Naturality of Flows***)**.** *Let $F : M \to N$, and let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be F-related vector fields that generate global flows θ and η respectively. Then for every $t \in \mathbb{R}$,the following diagram commutes*

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ {\scriptstyle \theta_t}\downarrow & & \downarrow{\scriptstyle \eta_t} \\ M & \xrightarrow[F]{} & N \end{array}$$

This also tells us how flows transform under diffeomorphism.

Corollary 2.7. *Let $F : M \to N$ be a diffeomorphism, and let $X \in \mathfrak{X}(M)$ with flow θ. Then the flow of the pushforward F_*X is $\eta_t = F \circ \theta_t \circ F^{-1}$.*

Given two vector fields V and W, we can ask how the vector field W changes along the flow of V. This is the notion of a Lie derivative.

Definition 2.8. Let $V, W \in \mathfrak{X}(M)$. Then the ***Lie derivative*** of W with respect to V is another vector field denoted $\mathcal{L}_V W$, defined by

$$\mathcal{L}_V W|_p = \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)})$$

where θ denotes the flow of W.

Luckily, computing the Lie derivative is easy!

Theorem 2.9.

$$\mathcal{L}_V W = [V, W]$$

We mentioned before that the fact that $[\partial_i, \partial_j] = 0$ is in some sense, the defining feature of the coordinate vector fields. It captures the idea that mixe partial derivates are the same, independent of the order of differentiation.

Theorem 2.10. *Suppose we have a collection $\{E_i\}$ of vector fields that form a local frame for $U \subset M$ and $[E_i, E_j] = 0$ for all i, j. Then there exist functions $x^i : U \to \mathbb{R}$ such that $E_i = \partial_i$.*

There's a higher dimensional analogue of vector fields and flows.

Definition 2.11. A ***a distribution*** D is a rank k subbundle of the tangent bundle TM, i.e. a smoothly varying family of k-dimensional subspaces of the tangent spaces. We denote the subspace over a point p by D_p.

From the definition, we see that a vector field is a special case of a distribution – it is a rank 1 distribution. With vector fields, we could integrate them into flows, giving a family of integral curves, i.e. 1-dimensional submanifolds of M whose tangent spaces are the span of the vector field. We can ask a similar question for general distributions.

Definition 2.12. Let D be a rank k distribution. Then a submanifold $N \subset M$ is an ***integral manifold*** of D if for each point p, we have that $T_pN = D_p$.

However, unlike vector fields, integral manifolds need not exist, even in small neighborhoods. A distribution D is said to be ***integrable*** if integral manifolds exist. The obstructure here is something called *involutivity*.

Definition 2.13. A distribution D is ***involutive*** if for any vector fields X, Y lying entirely in D (i.e. $X_p \in D_p$ and $Y_p \in D_p$ for all p), we have that $[X, Y]$ lies entirely in D

The intuition here is that if a distribution has an integral manifold, the restriction of the distribution to the integral manifolds is isomorphic to the tangent bundle of the integral manifold, so vector fields should be closed under the bracket. A theorem of Frobenius then tells us that this is a necessary and sufficient condition.

Theorem 2.14 (***The Frobenius Theorem***)**.** *A distribution D is integrable if and only if it is involutive.*