Math Fundamentals: Compactness

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Our topic of focus today is compactness. The model space you should have in your head is $[a, b] \subset \mathbb{R}$, a closed and bounded interval. Mathematicians grappled for a long time with what the definition of compactness should be, trying to find the right notion that captures some of the good properties of $[a, b]$. Some of these properties include:

- A continuous function $f : [a, b] \to \mathbb{R}$ has a maximum and a minimum. Moreover, f is uniformly continuous.
- Any bounded sequence in $[a, b]$ has a convergent subsequence.

What do these properties mean? You should be familiar with extrema of real-valued functions and have at least an intuitive notion of what continuity is. Uniform continuity is something we will discuss below.

Definition. *Let $f : X \to Y$ be a map between topological spaces. f is* ***continuous*** *if for every open subset $U \subset Y$, $f^{-1}(U)$ is open in X.*

For those of you who know some analysis, you have probably seen another notion of continuity. Namely, given a map $f : (X, d) \to (Y, \rho)$ of metric spaces, f is continuous at $x \in X$ if for every $\epsilon > 0$ there is $\delta > 0$ such that

$$d(x, x') < \delta \implies \rho(f(x), f(x')) < \epsilon.$$

In other words, as Prof. Freed likes to say, the images in Y of sufficiently close points in X to x look the same in our fuzzy vision that only lets us see things up to ϵ distance. f is continuous if it is continuous at every $x \in X$. This metric notion of continuity is equivalent to the above if we work with metric topologies.

Definition. *Let X be a topological space. An* ***open cover*** *of X is a collection $\mathcal{C} := \{X_\alpha\}_{\alpha \in J}$ of open subsets of X such that*

$$X \subset \bigcup_{\alpha \in J} U_\alpha.$$

A ***subcover*** *of $\mathcal{C}$ is just a subset of $\mathcal{C}$ that is an open cover of X. X is* ***compact*** *if every open cover of X has a finite subcover.*

The classic example of a non-compact space is the open interval $(0, 1) \subset \mathbb{R}$. Consider the open cover consisting of sets $U_n := (1/n, 1 - 1/n)$ with $n \geq 3$ an integer. How do we find compact subsets of $\mathbb{R}$ or, more generally, $\mathbb{R}^n$? Given a compact metric space X, it is always the case that X is closed and bounded. This condition is sufficient when $X = \mathbb{R}^n$.

Theorem (Heine-Borel)**.** *$A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

This already provides us with a wealth of useful compact spaces, especially the spheres S^n and the genus g tori.[1] This also allows us to see that the Cantor set is compact. Returning to the original motivation for compactness, things work out nicely.

Definition. *A topological space X is **limit point compact** if every infinite subset has a limit point. A metric space (X, d) is **sequentially compact** if every sequence in X has a convergent subsequence.*

Theorem (Bolzano-Weierstrass)**.** *Let X be a metric space. The following are equivalent:*

(i) *X is compact.*

(ii) *X is limit point compact.*

(iii) *X is sequentially compact.*

Given X a compact space, Y a topological space, and $f : X \to Y$ continuous, we have that $f(X)$ is compact. This implies:

Theorem (Extreme Value Theorem)**.** *Let X be a compact metric space and $f : X \to \mathbb{R}$ continuous. Then, f has a maximum and a minimum.*

It is not too hard to show that finite products of compact sets are compact. In fact:

Theorem (Tychonoff)**.** *An arbitrary product of compact spaces is compact.*

This theorem is equivalent to the axiom of choice and is extremely useful in functional analysis for proving results like the Banach-Alaoglu Theorem. Switching gears a bit, it turns out that the Heine-Borel Theorem has a nice generalization.

Definition. *Let (X, d) be a metric space. A sequence $\{x_n\}_{n\geq 1}$ in X is a **Cauchy sequence** if the terms of the sequence get arbitrarily close together – that is, given $\epsilon > 0$, there exists $N > 0$ such that*

$$m, n \geq N \implies d(x_m, x_n) < \epsilon.$$

*X is **complete** if every Cauchy sequence in X converges. X is **totally bounded** if, for every $\epsilon > 0$, there exist $x_1, \ldots, x_n \in X$ such that*

$$X \subset B_\epsilon(x_1) \cup \cdots B_\epsilon(x_n).$$

Theorem. *Let X be a metric space. Then, X is compact if and only if it is complete and totally bounded.*

This allows us to see that the unit ball is not compact for an infinite dimensional normed linear space. Compact spaces also feature prominently when working with infinite dimensional function spaces because they allow us to do approximation arguments.

Definition. *Let X be a compact space. Define $C(X)$ to be the set of continuous functions $X \to \mathbb{R}$.*

[1] You should think of a genus g torus as the surface of a donut with g holes in it.

This is a commutative $\mathbb{R}$-algebra under pointwise operations, which means that $C(X)$ is both a ring and an $\mathbb{R}$-vector space with

$$\begin{aligned}(fg)(x) &:= f(x)g(x),\\ (f+g)(x) &:= f(x)+g(x),\\ (rf)(x) &:= rf(x),\end{aligned}$$

for $r \in \mathbb{R}$ and $f, g \in C(X)$. This space has a natural norm $\|\cdot\|$ given by

$$\|f\| := \max_{x \in X} |f(x)|.$$

A sequence of functions $f_n : X \to \mathbb{R}$ is said to ***converge pointwise*** *to $f : X \to \mathbb{R}$ if $f_n(x) \to f(x)$ in $\mathbb{R}$ for every $x \in X$. The functions f_n* ***converge uniformly*** *to f if $f_n \to f$ in $C(X)$ with the above choice of norm.*

The space $C(X)$ is complete since the uniform limit of continuous functions is continuous.

Remark. *$C(X)$ is an extremely useful and interesting object that shows up in almost every field of math.*[2] *A smattering of interesting results include:*

(i) *Stone-Weierstrass Theorem*

(ii) *Serre-Swan Theorem*

(iii) *Gelfand-Neimark Theorem*

Theorem (Arzelà-Ascoli)**.** *Let (X, d) be a compact metric space. Then, $A \subset C(X)$ is precompact if and only if it is bounded and equicontinuous.*

Here, saying A is precompact means its closure in $C(X)$ is compact and saying that A is equicontinuous means that for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\max_{d(x,y)<\delta} |f(x) - f(y)| < \epsilon.$$

Definition. *Let $(X, d), (Y, \rho)$ be metric spaces. Then, $f : X \to Y$ is* ***uniformly continuous*** *if for every $\epsilon > 0$ there is $\delta > 0$ such that $d(x, x') < \delta \implies \rho(f(x), f(x')) < \epsilon$.*

Notice how the quantifiers differ from the case of ordinary continuity.

Theorem. *Let $f : X \to Y$ be a continuous map of metric spaces with X compact. Then, f is uniformly continuous.*

Uniform continuity is great because it affords us a lot of control over the behavior of functions. In particular, uniform convergence allows us to switch the order of many different kinds of limits processes. This is the basis for using what is called the compact-open topology in complex analysis. From this come cool results like Montel's Theorem and the Riemann Mapping Theorem.

[2] Often with the extra assumption that X is Hausdorff.