1. Compute $29 \cdot 31$ quickly in your head.

2. Take a 4×4 checkerboard and remove the upper-left and bottom-right squares. Can you cover the remaining squares with 2×1 tiles?

   Try generalizations of this puzzle to different shapes/sizes and removing different squares.

3. Estimate $\sum_{n=1}^{100} \sqrt{n}$

4. Say a whole number is a *happy number* if the sum of the squares of its digits is 1 or another happy number (e.g. 10 and 31 are happy numbers). Write an algorithm that finds all the happy numbers between 1 and 1000.

5. Write an algorithm that computes the number of distinct ways can you walk up a staircase with n steps if you can only walk up 1, 2, or 3 steps at a time. Assume order matters first. As a generalization, try the unordered case.

6. There are 100 closed lockers in a hallway. The principal begins by opening all 100 lockers. Next, she closes every second locker. Then, on her third pass, she toggles every third locker (closes it if it's open, opens it if it's closed). This process continues for 100 passes, such that on the ith pass, the principal toggles every ith locker. After the 100th pass in the hallway, in which she only toggles locker #100, how many lockers are open?

7. Suppose you are given a rectangle partitioned into subrectangles such that each subrectangle has either integer length or width. Prove that the big rectangle has integer length or width. What about the case where "integer" is replaced with "rational?"

8. Hang a picture frame using two nails such that if you pull out either nail the picture frame falls down. (Hint: the fundamental group)

9. You have 3 friends in Seattle. On any given day, there's a 50% chance that it's raining in Seattle. Your friends also like to mess with you so that when you call them to ask if it's raining in Seattle, each will lie to you with probability $\frac{1}{4}$. If you called up all 3 friends today and each told you that it's raining, what's the probability that it's actually raining in Seattle?

10. You have 13 weights such that if you remove any one weight, the remaining 12 can be split into two groups of 6 of equal total weight. Show that all 13 weights have equal weight.

11. We have 10 identical bottles of identical looking pills. Each bottle contains hundreds of pills. Of the 10 bottles, 9 bottles contain pills that weigh 1 gram, but 1 bottle contains pills that weigh 1.1 grams. Given a scale, how would you find the heavy bottle? You can use the scale only once.

    Generalizations: What if there are 100 bottles all but one of which contain 1 gram pills? What about countably infinitely bottles? What if all but two bottles contain 1 gram pills?

12. There is an ant on each vertex of an equilateral triangle. Each ant randomly picks a direction and begins moving along the sides of the triangle. What's the probability that there is a collision between two ants? Generalizations: what if "triangle" is replaced with "regular n-gon?" Consider also the case of ants walking along the edges of Platonic solids.

13. A jar contains 100 coins, 99 of which are fair and 1 which has heads on both sides. You pick a coin out of the jar at random and toss it 10 times. If you see heads on all 10 tosses, what's the probability that the next toss also yields heads?

14. You have 8 identical looking marbles and an old-timey balance scale. 7 of the marbles weigh the same but 1 of the marbles is slightly heavier. How would you find the heavy marble if you are only allowed to use the balance scale twice?

15. You have a bunch of marbles on a lattice (say $\mathbb{Z}^2$) and you can remove a marble if there's another with the same x or y coordinate as that one. What's the minimum number of marbles that can be left after these moves?