Lecture 1

Lie Groups and Lie Algebras

Lie Groups

Prior to defining a Lie group we begin by reiterating the definition of the two fundamental objects in mathematics necessary to construct them. The first is that of a group.

Definition (Group)**.** A group is a set G along with a binary operation $* : G \times G \to G$ and a distinguished element $e \in G$ such that the following three axioms hold:

(I) (Associative law) for every $a, b, c \in G$, $a * (b * c) = (a * b) * c$;

(II) (Identity) $e * a = a * e = a$ for all $a \in G$.

(III) (Inverse) for all $a \in G$ there exists an $b \in G$ such that $a * b = b * a = e$

The second definition is that of a smooth, or C^∞, manifold.

Definition (Manifold)**.** An n-dimensional (topological) manifold is a second countable, Hausdorff space M with some cover of open sets $\mathcal{U}$ and maps $\phi : U \to \mathbb{R}^n$ for each $U \in \mathcal{U}$ that are homeomorphisms onto their images (in particular, M is locally homeomorphic to $\mathbb{R}^n$). We commonly refer to the pairs (U, ϕ) as coordinate charts and call the collection

$$\mathcal{A} = \{(U, \phi) : (U, \phi) \text{ is a coordinate chart of } M\}$$

an atlas for M. If $(U_1, \phi_1), (U_2, \phi_2) \in \mathcal{A}$, we call the map $\psi_{12} : U_1 \cap U_2 \to U_1 \cap U_2$ defined by $\psi_{12} = \phi_2 \circ \phi_1^{-1}$ a transition map between the coordinate charts (U_1, ϕ_1) and (U_2, ϕ_2). If the set of all transition maps for $\mathcal{A}$ belong to some set $\mathcal{S}$ we say that the manifold M belongs to the structure class $\mathcal{S}$. In particular, a smooth manifold is a manifold M belonging to the structure class C^∞ of smooth functions (every atlas belongs to the structure class C^0 of continuous functions).

We note for any atlas $\mathcal{A}$ of a manifold M with structure class $\mathcal{S}$ there exists a unique maximal atlas on M containing $\mathcal{A}$ with structure class $\mathcal{S}$. In order to avoid ambiguity, we will always assume given a manifold M we have chosen a maximal atlas. We combine the two definitions into a third definition, that of a Lie group.

Definition (Lie group)**.** A group G is referred to as a Lie group if it admits the structure of a smooth manifold in a way compatible with the group operations. In particular, we require that with the given C^∞ structure the multiplication map and inverse maps

$$* : G \times G \to G$$

$$(-)^{-1} : G \to G$$

both be smooth.

We note since a Lie group belongs to both the category of groups and smooth manifolds a morphism between two Lie groups should also belong to both the category of groups and smooth manifolds. We are thus led to the following defintion for a Lie group homomorphism:

Definition (Lie group homomorphism)**.** We call a map between two Lie groups a Lie group homomorphism if it is both smooth and a group homomorphism. Thus, an isomorphism of Lie groups is both a diffeomorphism and an isomorphism of groups.

We would also like to define the notion of a Lie subgroup of a Lie group. Given a group G, we would like $H \subset G$ to respect both the group structure and the smooth structure of G. We thus have the following definition

Definition (Lie subgroup)**.** Let G be a Lie group and $H \subset G$. We call H a (closed) Lie subgroup if H is simultaneously a subgroup and closed submanifold of G. We call H an immersed subgroup if it is the image of some injective homomorphism into G.

We emphasize that every Lie subgroup of G is an immersed subgroup of G under the inclusion map but the converse need not be true. For a counterexample see page 94 of [FH91].

Examples of Lie groups

We now cite some examples of Lie groups, noting along the way which ones we will later be studying in depth. The simpliest example of a Lie group is simply the additive group $\mathbb{R}^n$. Similarly, the additive group $\mathbb{C}^n$ becomes a Lie group when viewed as a smooth manifold over $\mathbb{R}^{2n}$. Any finite group G can also be viewed as a zero dimensional Lie group when equipped with the discrete topology (that is, every singleton subset of G is declared to be open).

A more interesting example of a Lie group is the multiplicative group $\mathrm{GL}_n\mathbb{R}$ of $n \times n$ invertible real matrices. To see this we consider $\mathrm{GL}_n\mathbb{R}$ as embedded in $\mathcal{M}^{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ via inclusion. The determinant $\det : \mathcal{M}^{n\times n}(\mathbb{R}) \to \mathbb{R}$ is continuous (it is simply a multivariable polynomial) and the set $U = \mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$. It follows now that $\mathrm{GL}_n\mathbb{R} = \det^{-1}(U)$ is an open subset of $\mathcal{M}^{n\times n}(\mathbb{R})$ and thus is a submanifold of $\mathcal{M}^{n\times n}(\mathbb{R})$ with the subspace topology. Under this topology matrix multiplication and inversion smooth, and thus, $\mathrm{GL}_n\mathbb{R}$ is a Lie group. More abstractly, for a vector space V over a field $\mathbb{F}$ we define $\mathrm{GL}(V)$ (the general linear group of V) to be automorphism group of V. We note if V is n-dimensional, by choosing a basis we have an isomorphism $V \cong \mathbb{F}^n$ and thus $\mathrm{GL}(V) \cong \mathrm{GL}_n\mathbb{F}$ (it is standard to write $\mathrm{GL}(\mathbb{F}^n) = \mathrm{GL}_n\mathbb{F}$). We can repeat similar analysis to the case $\mathbb{F} = \mathbb{R}$ to show that $\mathrm{GL}_n\mathbb{C}$ is a Lie group. It thus follows that for all real and complex vector spaces that $\mathrm{GL}_n(V)$ is a Lie group.

We now define some very important Lie subgroups of $\mathrm{GL}_n\mathbb{R}$ and $\mathrm{GL}_n\mathbb{C}$. The first are the special linear groups $\mathrm{SL}_n\mathbb{R}$ and $\mathrm{SL}_n\mathbb{C}$ which are defined intrinsically to be automorphisms of $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively, preserving a volume form. Explicitly $\mathrm{SL}_n\mathbb{R}$ and $\mathrm{SL}_n\mathbb{C}$ are the matrices in $\mathrm{GL}_n\mathbb{R}$ and $\mathrm{GL}_n\mathbb{C}$, respectively, having determinant 1. The groups of real upper-triangular matrices B_n and real upper-triangle unipotent matrices (those having only 1's on the diagonal) N_n are also Lie subgroups of $\mathrm{GL}_n\mathbb{R}$.

A more interesting example of a Lie subgroup of $\mathrm{GL}_n\mathbb{R}$ is the orthogonal group $\mathrm{O}(n)$. This is defined to be the matrices preserving a symmetric, positive definite bilinear form $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Taking Q to be the standard inner product on $\mathbb{R}^n$ we can explicitly describe $\mathrm{O}(n)$ as the $A \in \mathrm{GL}_n\mathbb{R}$ such that $A^T = A^{-1}$. We note that for all $A \in \mathrm{O}(n)$ $\det A = \pm 1$. We define the special orthogonal group to be $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}_n\mathbb{R}$, that is those matrices in $\mathrm{O}(n)$ with determinant 1. If $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a symmetric, nondegenerate signature (p, q) bilinear form (that is, Q has k positive and l negative eigenvalues) we define the Lie groups $\mathrm{O}(p, q) \subset \mathrm{GL}_n\mathbb{R}$ to be the matrices preserving Q and $\mathrm{SO}(p, q) = \mathrm{O}(p, q) \cap \mathrm{SL}_n\mathbb{R}$ (note that we must have $k + l = n$). Finally, if $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a skew-symmetric, nondegenerate bilinear form we denote the Lie group of determinant 1 matrices preserving Q as $\mathrm{Sp}_n\mathbb{R}$ (note in this case n must be even).

We can repeat the above analysis for $\mathrm{GL}_n\mathbb{C}$ to define the Lie groups $\mathrm{SO}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$. We have, however, one more important Lie subgroup $\mathrm{U}(n)$ (the unitary group) consisting of matrices in $\mathrm{GL}_n\mathbb{C}$ preserving a positive definite Hermitian inner product (we call H a Hermitian form on $\mathbb{C}^n$ if $H(au, bv) = \overline{a}H(u, v)b$ and $H(u, v) = \overline{H(v, u)}$ for all $a, b \in \mathbb{C}$ and $u, v \in \mathbb{C}^n$). We can explicitly define $\mathrm{U}(n)$ to be the $A \in \mathrm{GL}_n\mathbb{C}$ such that $\overline{A}^T = A^{-1}$. We note for all $A \in \mathrm{U}(n)$ we have $|\det A| = 1$. We define the special unitary group to be $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}_n\mathbb{C}$.

We note that all of the examples given in this section are Lie subgroups of $\mathrm{GL}_n\mathbb{R}$ for big enough n. To see this for $\mathbb{R}^n$ and $\mathbb{C}^n$ consider the map into the general linear group given by

$$(x_1, \ldots, x_n) \mapsto \begin{pmatrix} e^{x_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{x_n} \end{pmatrix}$$

It is easy to verify the image is a Lie subgroup of the general linear group and the resulting map is a Lie group isomorphism. Similarly, in the case G is finite we can use Cayley's theorem to embed G as a subgroup of $S_{|G|}$ and then embed $S_{|G|}$ as a subgroup of $\mathrm{GL}_{|G|}\mathbb{R}$. The group $\mathrm{GL}_n\mathbb{C}$ also embeds as a subgroup of $\mathrm{GL}_{2n}\mathbb{R}$ under the identification

$$a + bi = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

While many Lie groups arise as Lie subgroups of a general linear group, it is important to note that not all do (for example, the universal cover of $\mathrm{SL}_n\mathbb{R}$).

Remark. The notation $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{SO}(p,q)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are reserved for the real groups described above. We note, however, that over any field we can describe many of these groups in terms of algebraic expressions. When we are referring to these algebraic groups we will use subscripts (e.g. GL_n, SL_n, SO_n, and Sp_{2n}).

Lie algebras

So far we have defined a Lie group and cited many examples, but have done little to explain what makes them any better to work with than the typical group. We begin first by proving a theorem for generic topological groups (groups admitting a topology such that multiplication and inversion are continuous) that motivates the proceeding discussion

Theorem. *Let G be a connected topological group and U a neighborhood of the identity. Then G is generated by U.*

Proof. Recall that G being connected means that there does not exists nonempty open sets $U, V \subset G$ such that $G = U \cup V$ and $U \cap V = \emptyset$. Now let U be a neighborhood of the identity. Without loss of generality we can assume U is closed under inversion, as the set $U^{-1} = \{u^{-1} : u \in U\}$ is open (since inversion is a homeomorphism in a topological group) so we simply shrink U to $U \cap U^{-1}$. We now define

$$W = \{g \in G : g = g_1 \cdots g_n \text{ for some } g_1, \ldots, g_n \in U\},$$

that is W is the set of elements in g generated by U. We claim that W is both open and closed. To see W is open, consider the set $gU = \{gu : u \in U\}$. $g \in gU$ since $e \in U$ and gU is open since left multiplication is a homeomorphism by the axioms of a topological group. If $g \in W$ then for all $u \in U$ it follows $gu \in W$ so $gU \subset W$. Thus for all $g \in W$ we have gU is an open neighborhood of g contained in W, and therefore, W is open. Now let $g \notin W$ we claim that $gU \cap W = \emptyset$. If not there is some $u \in U$ such that $gu \in W$. This implies $gu = g_1 \cdots g_n$ for some $g_1, \ldots, g_n \in U$. But then $g = g_1 \cdots g_n u^{-1}$ and since $u^{-1} \in U$ we conclude $g \in W$, a contradiction. It follows now that g is not a limit point of W, and thus, W must contain all of its limit points and is therefore closed. Since W is both open and closed it must be empty or all of G. If not we can write $G = W \cup W^c$ with W and W^c open, nonempty, and disjoint contradicting the connectedness of G. Since $e \in W$ it follows that $W = G$. ∎

Now let $\rho : G \to H$ be any homomorphism between two connected Lie groups. The above theorem implies that ρ is completely determined by what it does on any open set containing the identity of G (this can be restated as ρ is determined by its germ at $e \in G$). In fact, this leads one to hypothesize the following conjecture:

Conjecture. *Let G and H be Lie groups, with G connected. A map $\rho : G \to H$ is uniquely determined by its differential $d\rho_e : T_eG \to T_eH$ at the identity.*

We will see later that this conjecture turns out to be true, but prior we must develop some more theory.

The construction that follows is from Chapter 8 of [FH91]. We define $L_g : G \to G$ to simply be the map $L_g(h) = gh$ defined by left multiplication. In the case of a Lie group this map is a diffeomorphism with inverse $L_{g^{-1}}$. Now we can characterize a smooth map $\rho : G \to H$ as homomorphism if and only if the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ {\scriptstyle L_g}\downarrow & & \downarrow{\scriptstyle L_{\rho(g)}} \\ G & \xrightarrow[\rho]{} & H \end{array}$$

The problem with this characterization is that the differential does not fix the tangent space at any point since L_g has no fixed points unless $g = e$. We thus turn our attention to different maps, those given by conjugation. Explicitly for any $g \in G$ we will define the map

$$\Psi_g : G \to G \text{ by } \Psi_g(h) = ghg^{-1}.$$

We note that Ψ_g is an automorphism of g for all G, and thus, we have a map $\Psi : G \to \mathrm{Aut}(G)$ given by $g \mapsto \Psi_g$. The map Ψ is natural in the sense the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ {\scriptstyle \Psi_g}\downarrow & & \downarrow{\scriptstyle \Psi_{\rho(g)}} \\ G & \xrightarrow[\rho]{} & H \end{array}$$

commutes for all $g \in G$ and homomorphism $\rho : G \to H$. We can thus give the characterization that a *homomorphism ρ respects the action of a group G on itself by conjugation*.

We note that $\Psi_g(e) = e$ for all $g \in G$, that is Ψ_g fixes the identity element of G. We now set

$$\mathrm{Ad}_g = (d\Psi_g)_e : T_eG \to T_eG$$

This defines a map

$$\mathrm{Ad} : G \to \mathrm{Aut}(T_eG)$$

known as the adjoint representation of G. We see that by differentiating the previous diagram we obtain

$$\begin{array}{ccc} T_eG & \xrightarrow{d\rho_e} & T_eH \\ {\scriptstyle \mathrm{Ad}_g}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_{\rho(g)}} \\ T_eG & \xrightarrow[d\rho_e]{} & T_eH \end{array}$$

commutes. That is, we have for any $v \in T_eG$ that

$$d\rho_e(\mathrm{Ad}_g(v)) = \mathrm{Ad}_{\rho(g)}(d\rho_e(v)).$$

We thus get the implication that *a homomorphism ρ respects the adjoint action of a group G on on its tangent space T_eG at the identity*.

While the above result is useful, it still involves the use of the homomorphism ρ. We rectify this by taking the derivative of Ad at the identity, noting that since $\text{Aut}(T_eG) \subset \text{End}(T_eG)$ (the endomorphisms of T_eG, that is, linear maps $T_eG \to T_eG$) is open there is a natural identification of $T_e\text{Aut}(T_eG)$ with $\text{End}(T_eG)$. This gives us a map

$$d(\text{Ad})_e = \text{ad} : T_eG \to \text{End}(T_eG).$$

We can view ad as a map taking 2 inputs in $X, Y \in T_eG$ and giving one output $\text{ad}_X(Y) \in T_eG$. In particular, under this identification ad becomes a bilinear map

$$T_eG \times T_eG \to T_eG.$$

We now define

$$[X, Y] = \text{ad}_X(Y).$$

We now note that the following diagram commutes:

$$\begin{array}{ccc} T_eG & \xrightarrow{d\rho_e} & T_eH \\ \Big\downarrow{\scriptstyle \text{ad}_X} & & \Big\downarrow{\scriptstyle \text{ad}_{d\rho_e(X)}} \\ T_eG & \xrightarrow[d\rho_e]{} & T_eH \end{array}$$

that is,

$$dp_e([X, Y]) = [dp_e(X), dp_e(Y)] \text{ for all } X, Y \in T_eG.$$

We get the characterization that *the differential $d\rho_e$ of a homomorphism ρ on a Lie group G respects the adjoint action of the tangent space to G on itself*.

While the bracket operation $[-,-] : T_eG \times T_eG \to T_eG$ seems fairly abstract, there are many cases in which it can be made fairly explicit. For instance, take $G = \text{GL}_n\mathbb{R}$. For any general matrix $X \in \mathbb{R}^{n\times n}$ we can find a smooth arc $\gamma : [0, 1] \to G$ such that $\gamma(0) = I$ and $\gamma'(0) = X$ (we will elaborate on this next lecture when we define the exponential map). It follows then that $T_eG = \mathbb{R}^{n\times n}$ and

$$\text{Ad}_g(X) = \frac{d}{dt}\Psi_g(\gamma(t))\bigg|_{t=0} = \frac{d}{dt}(g\gamma(t)g^{-1})\bigg|_{t=0} = gXg^{-1},$$

in particular the adjoint action of G on T_eG is simply conjugation by elements of G. Letting $X, Y \in T_eG$ and $\gamma : [0, 1] \to G$ be as before we can compute using the product rule that

$$\begin{aligned} [X, Y] = \text{ad}_X(Y) &= \frac{d}{dt}\text{Ad}_{\gamma(t)}(Y)\bigg|_{t=0} \\ &= \frac{d}{dt}(\gamma(t)Y\gamma(t)^{-1})\bigg|_{t=0} \\ &= \gamma'(0)Y\gamma(0)^{-1} + \gamma(0)Y(-\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1}) \\ &= XY - YX. \end{aligned}$$

Thus, the bracket operation on the tangent space to $\text{GL}_n\mathbb{R}$ is simply the commutator (hence the notation).

Remark. We provide a quick proof of why $(\gamma(t)^{-1})' = -\gamma(t)^{-1}\gamma'(t)\gamma(t)^{-1}$. Note that for all $t \in [0, 1]$ we have $I = \gamma(t)\gamma(t)^{-1}$. Using the product rule, we can differentiate both sides to obtain

$$0 = \gamma'(t)\gamma(t)^{-1} + \gamma(t)(\gamma(t)^{-1})' \implies \gamma(t)(\gamma(t)^{-1})' = -\gamma'(t)\gamma(t)^{-1}.$$

We can now multiply both sides by $\gamma(t)^{-1}$ to obtain the desired identity $(\gamma(t)^{-1})' = -\gamma(t)^{-1}\gamma'(t)\gamma(t)^{-1}$.

In the next lecture we will prove for a general Lie group G that given any $X \in T_eG$ there exists a homomorphism $\rho : \mathbb{R} \to G$ satisfying $\rho(0) = e$ and $\rho'(0) = X$ (the so-called one parameter subgroups of G). We will use the existence of such homomorphisms to prove the following Theorem:

Theorem. *Let G be a Lie group and let $\mathfrak{g} = T_eG$. Then the adjoint action $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfies the following properties:*

1. *Skew-commutatitivity: $[X, Y] = -[Y, X]$ for any $X, Y \in \mathfrak{g}$;*
2. *The Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for any $X, Y, Z \in \mathfrak{g}$.*

Proof. We begin by proving (1). This is equivalent to the assertion that $[X, X] = 0$ for all $X \in \mathfrak{g}$, as assuming said assertion for any $X, Y \in \mathfrak{g}$ we can deduce

$$\begin{aligned} 0 &= [X+Y, X+Y] \\ &= [X, X] + [X, Y] + [Y, X] + [Y, Y] \\ &= [X, Y] + [Y, X] \\ &\Longrightarrow [X, Y] = -[Y, X]. \end{aligned}$$

We thus proceed to show $[X, X] = 0$ for all $X \in \mathfrak{g}$. Let $\rho : \mathbb{R} \to G$ satisfying $\rho(0) = e$ and $\rho'(0) = X$. We can then compute

$$\begin{aligned} [X, X] &= \frac{d}{dt}\mathrm{Ad}_{\rho(t)}(X)\Big|_{t=0} = \frac{d}{dt}\left(\frac{d}{ds}\Psi_{\rho(t)}(\rho(s))\Big|_{s=0}\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\frac{d}{ds}\rho(t)\rho(s)\rho(t)^{-1}\Big|_{s=0}\right)\Big|_{t=0} = \frac{d}{dt}\left(\frac{d}{ds}\rho(t)\rho(s)\rho(-t)\Big|_{s=0}\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\frac{d}{ds}\rho(t+s-t)\Big|_{s=0}\right)\Big|_{t=0} = \frac{d}{dt}\left(\frac{d}{ds}\rho(s)\Big|_{s=0}\right)\Big|_{t=0} = \frac{d}{dt}X\Big|_{t=0} = 0. \end{aligned}$$

This completes the proof of (1).

In order to prove (2), we let $\rho : \mathbb{R} \to G$ be a homomorphism satisfying $\rho(0) = e$ and $\rho'(0) = Z$. We then can compute

$$\begin{aligned} [Z, [X, Y]] &= \frac{d}{dt}\mathrm{Ad}_{\rho(t)}([X, Y])\Big|_{t=0} = \frac{d}{dt}d(\Psi_{\rho(t)})_e([X, Y])\Big|_{t=0} \\ &= \frac{d}{dt}\left[d(\Psi_{\rho(t)})_e(X), d(\Psi_{\rho(t)})_e(Y)\right]\Big|_{t=0} = \frac{d}{dt}\left[\mathrm{Ad}_{\rho(t)}(X), \mathrm{Ad}_{\rho(t)}(Y)\right]\Big|_{t=0} \\ &= \left[\frac{d}{dt}\mathrm{Ad}_{\rho(t)}(X)\Big|_{t=0}, Y\right] + \left[X, \frac{d}{dt}\mathrm{Ad}_{\rho(t)}(Y)\Big|_{t=0}\right] \\ &= [[Z, X], Y] + [X, [Z, Y]]. \end{aligned}$$

Using (1) we can rearrange the above to obtain $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, proving (2). ■

Using the above theorem as inspiration we are led to the following definition:

Definition (Lie algebra)**.** A Lie algebra $\mathfrak{g}$ is a vector space along with a skew-symmetric bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity. The bilinear map is commonly referred to as a Lie bracket on $\mathfrak{g}$. Given two Lie algebras $\mathfrak{g}, \mathfrak{h}$ a linear map $T : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if it respects the Lie brackets, that is if

$$T([X, Y]_{\mathfrak{g}}) = [T(X), T(Y)]_{\mathfrak{h}} \quad \text{for all } X, Y \in \mathfrak{g}.$$

If G is a Lie group then $\mathfrak{g} = T_eG$ becomes a Lie algebra with Lie bracket given by the adjoint action of the tangent space on itself. In this case $\mathfrak{g}$ is known as the Lie algebra of G.

Examples of Lie algebras

Before giving examples of Lie algebras we make a remark about notation. From this point forward the Lie algebra of a Lie group G will be denoted by the fraktur character $\mathfrak{g}$. For instance the Lie algebra of $\mathrm{GL}_n\mathbb{R}$ is commonly denoted as $\mathfrak{gl}_n\mathbb{R}$ and even more generally the Lie algebra of $\mathrm{GL}(V)$ is denoted as $\mathfrak{gl}(V)$. We showed above that as a vector space $\mathfrak{gl}_n\mathbb{R} \cong \mathbb{R}^{n\times n}$ (and more generally $\mathfrak{gl}(V) \cong \mathrm{End}(V)$), however when we refer to a Lie algebra $\mathfrak{g}$ for a known Lie group G we are always referring to the underlying vector space *along* with its inherited Lie bracket.

The first Lie group we will consider is $\mathrm{SL}_n\mathbb{R}$. Recall that for all $X \in \mathrm{SL}_n\mathbb{R}$ we have $\det(X) = 1$. We will use the following Lemma to find $\mathfrak{sl}_n\mathbb{R}$.

Lemma. *Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ be differentiable and $\gamma(0) = I$. Then*

$$\frac{d}{dt}\det(\gamma(t))\bigg|_{t=0} = Tr\left(\gamma'(0)\right).$$

Proof. Let $\gamma_{ij}(t)$ denote the (i,j) entry of $\gamma(t)$ and let $\gamma_{\hat{i}\hat{j}}(t)$ denote $\gamma(t)$ without the i^{th} row and j^{th} column. We can then write that

$$\det(\gamma(t)) = \sum_{j=1}^{n}(-1)^{j+1}\gamma_{1j}(t)\det(\gamma_{\hat{1}\hat{j}}(t)).$$

Taking the derivative of both sides at zero we obtain

$$\begin{aligned}
\frac{d}{dt}\det(\gamma(t))\bigg|_{t=0} &= \frac{d}{dt}\left[\sum_{j=1}^{n}(-1)^{j+1}\gamma_{1j}(t)\det(\gamma_{\hat{1}\hat{j}}(t))\right]\Bigg|_{t=0} \\
&= \sum_{j=1}^{n}\left[(-1)^{j+1}\gamma'_{1j}(t)\det(\gamma_{\hat{1}\hat{j}}(0)) + (-1)^{j+1}\gamma_{1j}(0)\,\frac{d}{dt}\det(\gamma_{\hat{1}\hat{j}}(t))\bigg|_{t=0}\right] \\
&= \gamma'_{11}(0) + \frac{d}{dt}\det(\gamma_{\hat{1}\hat{1}}(t))\bigg|_{t=0}
\end{aligned}$$

where the final equality uses the fact $\gamma(0) = I$. Repeating recursively we see that

$$\frac{d}{dt}\det(\gamma(t))\bigg|_{t=0} = \gamma'_{11}(0) + \gamma'_{22}(0) + \cdots + \gamma'_{nn}(0) = \mathrm{Tr}\left(\gamma'(0)\right)$$

as claimed. ■

Now assume $X \in \mathfrak{sl}_n\mathbb{R}$ and let $\rho_X : \mathbb{R} \to \mathrm{SL}_n\mathbb{R}$ be the corresponding homomorphism with $\rho_X(0) = I$ and $\rho'_X(0) = X$. Since $\det(\rho(t)) = 1$ for all $t \in \mathbb{R}$ the above Lemma implies $\mathrm{Tr}(X) = 0$. We see that the matrices in $\mathbb{R}^{n\times n}$ with trace 0 form an $n^2 - 1$ dimensional of $\mathfrak{sl}_n\mathbb{R}$, and since $\mathrm{SL}_n\mathbb{R}$ is $n^2 - 1$ dimensional, $\mathfrak{sl}_n\mathbb{R} = \{X \in \mathfrak{gl}_n\mathbb{R} : \mathrm{Tr}(X) = 0\}$ with Lie bracket being the commutator. We note that in the above analysis we could of chosen $\mathbb{C}$ instead and none of the computation would of changed.

We now consider the Lie groups defined by preservation of a bilinear form Q. If G is such a Lie group, $X \in \mathfrak{g}$ and let $\rho_X : \mathbb{R} \to G$ the corresponding homomorphism with $\rho_X(0) = I$ and $\rho'_X(0) = X$ we then have that

$$Q(\rho(t)v, \rho(t)w) = Q(v, w).$$

Deriving both sides with respect to t at 0 we obtain

$$Q(Xv, w) + Q(v, Xw) = 0.$$

If Q has a matrix representation M this is equivalent to the condition

$$X^T M + MX = 0.$$

If $G = \mathrm{O}(n)$ or $G = \mathrm{SO}_n\mathbb{R}$ the standard inner product on $\mathbb{R}^n$ has matrix representation I, so by comparing dimensions we see

$$\mathfrak{o}(n) = \mathfrak{so}_n\mathbb{R} = \left\{ X \in \mathbb{R}^{n\times n} : X^T = -X \right\},$$

that is both $\mathfrak{o}(n)$ and $\mathfrak{so}_n\mathbb{R}$ are the skew symmetric $n \times n$ matrices. If $G = \mathrm{Sp}_{2n}\mathbb{R}$ we take M to be the matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

representing the standard skew-symmetric, nondegenerate bilinear form on $\mathbb{R}^{2n}$. In this case comparing dimensions gives

$$\mathfrak{sp}_{2n}\mathbb{R} = \left\{ X \in \mathbb{R}^{n\times n} : X^T\Omega + \Omega X = 0 \right\}.$$

Left-invariant vector fields and the exponential map

We now finish the lecture by discussing something seemingly unrelated to everything previously which we use to show the existence of the homomorphisms $\rho : \mathbb{R} \to G$ satisfying $\rho(0) = e$ and $\rho'(0) = X$ for any $X \in \mathfrak{g}$. Let $v : M \to TM$ be a vector field, i.e. a smooth map into the tangent bundle such that if $\pi : TM \to M$ is the projection map we have $\pi \circ v = \mathrm{Id}_M$ (that is, v is a section of TM). Now assume that some Lie group G has a smooth left action on M. Then for each $g \in G$ we obtain a diffeomorphism $L_g : M \to M$ given by $m \mapsto gm$. We say that v is a left-invariant vector field if $d(L_g)_e(v(m)) = v(gm)$ for all $g \in G$ and $m \in M$.

In particular consider a Lie group G acting on itself via left multiplication. We can construct a left-invariant vector field on G simply by choosing some $X \in \mathfrak{g}$ and setting $v_X(g) = d(L_g)_e(X)$. It is simply to see that conversely, any left-invariant vector field on G by this action is uniquely determined by its value $v(e) \in \mathfrak{g}$. It turns out that that the bracket of two left-invariant vector fields is once again left-invariant (where the bracket is the standard bracket on vector fields given by the "commutator") and satisfies the axioms of a Lie algebra. In fact, while we won't prove it, the Lie algebra $\mathfrak{g}$ obtained by differentiating the conjugation automorphism and the Lie algebra obtained from left-invariant vector fields on G under the action of left multiplication are isomorphic.

Now choose $X \in \mathfrak{g}$ and take the left-invariant vector field $v_X(g) = d(L_g)_e(X)$. A basic theorem from differential equations allows us to find some integral curve $\rho : (-\varepsilon, \varepsilon) \to G$ with $\rho(0) = e$ for small $\varepsilon > 0$ such that ρ is uniquely determined by the condition

$$\rho'(t) = v_X(\rho(t)).$$

We claim this ρ is a homomorphism on this interval, that is $\rho(s+t) = \rho(s)\rho(t)$. Let $\alpha(t) = \rho(s+t)$ and $\beta(t) = \rho(s)\rho(t)$. It is easy to check $\alpha'(t) = v_X(\alpha(t))$ and $\beta'(t) = v_X(\beta(t))$ using the left-invariance of v_X. It follows then that $\alpha(t)$ and $\beta(t)$ are equal since $\alpha(0) = \beta(0)$ and integral curves are unique. We can now extend ρ to all of $\mathbb{R}$ by defining $\rho(s+t) = \rho(s)\rho(t)$ if ρ is defined at $s, t \in \mathbb{R}$ and not $s + t \in \mathbb{R}$.

The ρ we have constructed are the unique homomorphisms asserted to exist previously. Commonly we refer to such ρ as one-parameter subgroups of G and for $X \in \mathfrak{g}$ we denote

$$\rho(t) = \exp(tX) = e^{tX}.$$

Fixing $t = 1$ we obtain a map $\exp : \mathfrak{g} \to G$ defined by $X \mapsto e^X$. This map is referred to as the exponential map and gives us a gadget for taking elements of a Lie algebra and recovering elements of a Lie group. We will go into more detail on $\exp$ in the next lecture.