UT MUST: A Mini Course in Representation Theory

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Contents

Introduction

These notes act as a supplement to a 4 day lecture series given to the math club at the University of Texas at Austin held in the spring of 2019. The goal of these lectures is to introduce a beginner to the basic concepts fundamental to the representation theory of Lie groups and Lie algebras in finite dimensions. We in particular aim to define Lie groups and Lie algebras, show the correspondence between representations of a Lie algebra and that of the Lie group from which it arises, and find the irreducible finite dimensional representations of the Lie algebras $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$. For the abstract algebra used in the lectures we recommend using [DF04] as a reference and for the theory of smooth manifolds we recommend using [Lee03]. Whenever we need to use theorems from algebraic topology we will cite the proofs from [Hat02]. For those interested in learning more about representation theory of Lie groups we recommend [FH91] and [Bum04] as resources. We mainly use [FH91] Chapters 7-9,11, and 12 as references in these notes.

Lecture 1

Lie Groups and Lie Algebras

Lie Groups

Prior to defining a Lie group we begin by reiterating the definition of the two fundamental objects in mathematics necessary to construct them. The first is that of a group.

> **Definition** (Group)**.** A group is a set G along with a binary operation $* : G \times G \to G$ and a distinguished element $e \in G$ such that the following three axioms hold:
>
> (I) (Associative law) for every $a, b, c \in G$, $a * (b * c) = (a * b) * c$;
>
> (II) (Identity) $e * a = a * e = a$ for all $a \in G$.
>
> (III) (Inverse) for all $a \in G$ there exists an $b \in G$ such that $a * b = b * a = e$

The second definition is that of a smooth, or C^∞, manifold.

> **Definition** (Manifold)**.** An n-dimensional (topological) manifold is a second countable, Hausdorff space M with some cover of open sets $\mathcal{U}$ and maps $\phi : U \to \mathbb{R}^n$ for each $U \in \mathcal{U}$ that are homeomorphisms onto their images (in particular, M is locally homeomorphic to $\mathbb{R}^n$). We commonly refer to the pairs (U, ϕ) as coordinate charts and call the collection
>
> $$\mathcal{A} = \{(U, \phi) : (U, \phi) \text{ is a coordinate chart of } M\}$$
>
> an atlas for M. If $(U_1, \phi_1), (U_2, \phi_2) \in \mathcal{A}$, we call the map $\psi_{12} : U_1 \cap U_2 \to U_1 \cap U_2$ defined by $\psi_{12} = \phi_2 \circ \phi_1^{-1}$ a transition map between the coordinate charts (U_1, ϕ_1) and (U_2, ϕ_2). If the set of all transition maps for $\mathcal{A}$ belong to some set $\mathcal{S}$ we say that the manifold M belongs to the structure class $\mathcal{S}$. In particular, a smooth manifold is a manifold M belonging to the structure class C^∞ of smooth functions (every atlas belongs to the structure class C^0 of continuous functions).

We note for any atlas $\mathcal{A}$ of a manifold M with structure class $\mathcal{S}$ there exists a unique maximal atlas on M containing $\mathcal{A}$ with structure class $\mathcal{S}$. In order to avoid ambiguity, we will always assume given a manifold M we have chosen a maximal atlas. We combine the two definitions into a third definition, that of a Lie group.

> **Definition** (Lie group)**.** A group G is referred to as a Lie group if it admits the structure of a smooth manifold in a way compatible with the group operations. In particular, we require that with the given C^∞ structure the multiplication map and inverse maps
>
> $$* : G \times G \to G$$
>
> $$(-)^{-1} : G \to G$$
>
> both be smooth.

We note since a Lie group belongs to both the category of groups and smooth manifolds a morphism between two Lie groups should also belong to both the category of groups and smooth manifolds. We are thus led to the following defintion for a Lie group homomorphism:

> **Definition** (Lie group homomorphism)**.** We call a map between two Lie groups a Lie group homomorphism if it is both smooth and a group homomorphism. Thus, an isomorphism of Lie groups is both a diffeomorphism and an isomorphism of groups.

We would also like to define the notion of a Lie subgroup of a Lie group. Given a group G, we would like $H \subset G$ to respect both the group structure and the smooth structure of G. We thus have the following definition

> **Definition** (Lie subgroup)**.** Let G be a Lie group and $H \subset G$. We call H a (closed) Lie subgroup if H is simultaneously a subgroup and closed submanifold of G. We call H an immersed subgroup if it is the image of some injective homomorphism into G.

We emphasize that every Lie subgroup of G is an immersed subgroup of G under the inclusion map but the converse need not be true. For a counterexample see page 94 of [FH91].

Examples of Lie groups

We now cite some examples of Lie groups, noting along the way which ones we will later be studying in depth. The simpliest example of a Lie group is simply the additive group $\mathbb{R}^n$. Similarly, the additive group $\mathbb{C}^n$ becomes a Lie group when viewed as a smooth manifold over $\mathbb{R}^{2n}$. Any finite group G can also be viewed as a zero dimensional Lie group when equipped with the discrete topology (that is, every singleton subset of G is declared to be open).

A more interesting example of a Lie group is the multiplicative group $\mathrm{GL}_n\mathbb{R}$ of $n \times n$ invertible real matrices. To see this we consider $\mathrm{GL}_n\mathbb{R}$ as embedded in $\mathcal{M}^{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$ via inclusion. The determinant $\det : \mathcal{M}^{n\times n}(\mathbb{R}) \to \mathbb{R}$ is continuous (it is simply a multivariable polynomial) and the set $U = \mathbb{R} \setminus \{0\}$ is open in $\mathbb{R}$. It follows now that $\mathrm{GL}_n\mathbb{R} = \det^{-1}(U)$ is an open subset of $\mathcal{M}^{n\times n}(\mathbb{R})$ and thus is a submanifold of $\mathcal{M}^{n\times n}(\mathbb{R})$ with the subspace topology. Under this topology matrix multiplication and inversion smooth, and thus, $\mathrm{GL}_n\mathbb{R}$ is a Lie group. More abstractly, for a vector space V over a field $\mathbb{F}$ we define $\mathrm{GL}(V)$ (the general linear group of V) to be automorphism group of V. We note if V is n-dimensional, by choosing a basis we have an isomorphism $V \cong \mathbb{F}^n$ and thus $\mathrm{GL}(V) \cong \mathrm{GL}_n\mathbb{F}$ (it is standard to write $\mathrm{GL}(\mathbb{F}^n) = \mathrm{GL}_n\mathbb{F}$). We can repeat similar analysis to the case $\mathbb{F} = \mathbb{R}$ to show that $\mathrm{GL}_n\mathbb{C}$ is a Lie group. It thus follows that for all real and complex vector spaces that $\mathrm{GL}_n(V)$ is a Lie group.

We now define some very important Lie subgroups of $\mathrm{GL}_n\mathbb{R}$ and $\mathrm{GL}_n\mathbb{C}$. The first are the special linear groups $\mathrm{SL}_n\mathbb{R}$ and $\mathrm{SL}_n\mathbb{C}$ which are defined intrinsically to be automorphisms of $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively, preserving a volume form. Explicitly $\mathrm{SL}_n\mathbb{R}$ and $\mathrm{SL}_n\mathbb{C}$ are the matrices in $\mathrm{GL}_n\mathbb{R}$ and $\mathrm{GL}_n\mathbb{C}$, respectively, having determinant 1. The groups of real upper-triangular matrices B_n and real upper-triangle unipotent matrices (those having only 1's on the diagonal) N_n are also Lie subgroups of $\mathrm{GL}_n\mathbb{R}$.

A more interesting example of a Lie subgroup of $\mathrm{GL}_n\mathbb{R}$ is the orthogonal group $\mathrm{O}(n)$. This is defined to be the matrices preserving a symmetric, positive definite bilinear form $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Taking Q to be the standard inner product on $\mathbb{R}^n$ we can explicitly describe $\mathrm{O}(n)$ as the $A \in \mathrm{GL}_n\mathbb{R}$ such that $A^T = A^{-1}$. We note that for all $A \in \mathrm{O}(n)$ $\det A = \pm 1$. We define the special orthogonal group to be $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}_n\mathbb{R}$, that is those matrices in $\mathrm{O}(n)$ with determinant 1. If $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a symmetric, nondegenerate signature (p, q) bilinear form (that is, Q has k positive and l negative eigenvalues) we define the Lie groups $\mathrm{O}(p, q) \subset \mathrm{GL}_n\mathbb{R}$ to be the matrices preserving Q and $\mathrm{SO}(p, q) = \mathrm{O}(p, q) \cap \mathrm{SL}_n\mathbb{R}$ (note that we must have $k + l = n$). Finally, if $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a skew-symmetric, nondegenerate bilinear form we denote the Lie group of determinant 1 matrices preserving Q as $\mathrm{Sp}_n\mathbb{R}$ (note in this case n must be even).

We can repeat the above analysis for $\mathrm{GL}_n\mathbb{C}$ to define the Lie groups $\mathrm{SO}_n\mathbb{C}$ and $\mathrm{Sp}_{2n}\mathbb{C}$. We have, however, one more important Lie subgroup $\mathrm{U}(n)$ (the unitary group) consisting of matrices in $\mathrm{GL}_n\mathbb{C}$ preserving a positive definite Hermitian inner product (we call H a Hermitian form on $\mathbb{C}^n$ if $H(au, bv) = \overline{a}H(u, v)b$ and $H(u, v) = \overline{H(v, u)}$ for all $a, b \in \mathbb{C}$ and $u, v \in \mathbb{C}^n$). We can explicitly define $\mathrm{U}(n)$ to be the $A \in \mathrm{GL}_n\mathbb{C}$ such that $\overline{A}^T = A^{-1}$. We note for all $A \in \mathrm{U}(n)$ we have $|\det A| = 1$. We define the special unitary group to be $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}_n\mathbb{C}$.

We note that all of the examples given in this section are Lie subgroups of $\mathrm{GL}_n\mathbb{R}$ for big enough n. To see this for $\mathbb{R}^n$ and $\mathbb{C}^n$ consider the map into the general linear group given by

$$(x_1, \ldots, x_n) \mapsto \begin{pmatrix} e^{x_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{x_n} \end{pmatrix}$$

It is easy to verify the image is a Lie subgroup of the general linear group and the resulting map is a Lie group isomorphism. Similarly, in the case G is finite we can use Cayley's theorem to embed G as a subgroup of $S_{|G|}$ and then embed $S_{|G|}$ as a subgroup of $\mathrm{GL}_{|G|}\mathbb{R}$. The group $\mathrm{GL}_n\mathbb{C}$ also embeds as a subgroup of $\mathrm{GL}_{2n}\mathbb{R}$ under the identification

$$a + bi = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

While many Lie groups arise as Lie subgroups of a general linear group, it is important to note that not all do (for example, the universal cover of $\mathrm{SL}_n\mathbb{R}$).

Remark. The notation $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{SO}(p,q)$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are reserved for the real groups described above. We note, however, that over any field we can describe many of these groups in terms of algebraic expressions. When we are referring to these algebraic groups we will use subscripts (e.g. GL_n, SL_n, SO_n, and Sp_{2n}).

Lie algebras

So far we have defined a Lie group and cited many examples, but have done little to explain what makes them any better to work with than the typical group. We begin first by proving a theorem for generic topological groups (groups admitting a topology such that multiplication and inversion are continuous) that motivates the proceeding discussion

Theorem. *Let G be a connected topological group and U a neighborhood of the identity. Then G is generated by U.*

Proof. Recall that G being connected means that there does not exists nonempty open sets $U, V \subset G$ such that $G = U \cup V$ and $U \cap V = \emptyset$. Now let U be a neighborhood of the identity. Without loss of generality we can assume U is closed under inversion, as the set $U^{-1} = \left\{u^{-1} : u \in U\right\}$ is open (since inversion is a homeomorphism in a topological group) so we simply shrink U to $U \cap U^{-1}$. We now define

$$W = \{g \in G : g = g_1 \cdots g_n \text{ for some } g_1, \ldots, g_n \in U\},$$

that is W is the set of elements in g generated by U. We claim that W is both open and closed. To see W is open, consider the set $gU = \{gu : u \in U\}$. $g \in gU$ since $e \in U$ and gU is open since left multiplication is a homeomorphism by the axioms of a topological group. If $g \in W$ then for all $u \in U$ it follows $gu \in W$ so $gU \subset W$. Thus for all $g \in W$ we have gU is an open neighborhood of g contained in W, and therefore, W is open. Now let $g \notin W$ we claim that $gU \cap W = \emptyset$. If not there is some $u \in U$ such that $gu \in W$. This implies $gu = g_1 \cdots g_n$ for some $g_1, \ldots, g_n \in U$. But then $g = g_1 \cdots g_n u^{-1}$ and since $u^{-1} \in U$ we conclude $g \in W$, a contradiction. It follows now that g is not a limit point of W, and thus, W must contain all of its limit points and is therefore closed. Since W is both open and closed it must be empty or all of G. If not we can write $G = W \cup W^c$ with W and W^c open, nonempty, and disjoint contradicting the connectedness of G. Since $e \in W$ it follows that $W = G$. ■

Now let $\rho : G \to H$ be any homomorphism between two connected Lie groups. The above theorem implies that ρ is completely determined by what it does on any open set containing the identity of G (this can be restated as ρ is determined by its germ at $e \in G$). In fact, this leads one to hypothesize the following conjecture:

Conjecture. *Let G and H be Lie groups, with G connected. A map $\rho : G \to H$ is uniquely determined by its differential $d\rho_e : T_eG \to T_eH$ at the identity.*

We will see later that this conjecture turns out to be true, but prior we must develop some more theory.

The construction that follows is from Chapter 8 of [FH91]. We define $L_g : G \to G$ to simply be the map $L_g(h) = gh$ defined by left multiplication. In the case of a Lie group this map is a diffeomorphism with inverse $L_{g^{-1}}$. Now we can characterize a smooth map $\rho : G \to H$ as homomorphism if and only if the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ {\scriptstyle L_g}\downarrow & & \downarrow{\scriptstyle L_{\rho(g)}} \\ G & \xrightarrow[\rho]{} & H \end{array}$$

The problem with this characterization is that the differential does not fix the tangent space at any point since L_g has no fixed points unless $g = e$. We thus turn our attention to different maps, those given by conjugation. Explicitly for any $g \in G$ we will define the map

$$\Psi_g : G \to G \text{ by } \Psi_g(h) = ghg^{-1}.$$

We note that Ψ_g is an automorphism of g for all G, and thus, we have a map $\Psi : G \to \mathrm{Aut}(G)$ given by $g \mapsto \Psi_g$. The map Ψ is natural in the sense the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H \\ {\scriptstyle \Psi_g}\downarrow & & \downarrow{\scriptstyle \Psi_{\rho(g)}} \\ G & \xrightarrow[\rho]{} & H \end{array}$$

commutes for all $g \in G$ and homomorphism $\rho : G \to H$. We can thus give the characterization that a *homomorphism ρ respects the action of a group G on itself by conjugation*.

We note that $\Psi_g(e) = e$ for all $g \in G$, that is Ψ_g fixes the identity element of G. We now set

$$\mathrm{Ad}_g = (d\Psi_g)_e : T_eG \to T_eG$$

This defines a map

$$\mathrm{Ad} : G \to \mathrm{Aut}(T_eG)$$

known as the adjoint representation of G. We see that by differentiating the previous diagram we obtain

$$\begin{array}{ccc} T_eG & \xrightarrow{d\rho_e} & T_eH \\ {\scriptstyle \mathrm{Ad}_g}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_{\rho(g)}} \\ T_eG & \xrightarrow[d\rho_e]{} & T_eH \end{array}$$

commutes. That is, we have for any $v \in T_eG$ that

$$d\rho_e(\mathrm{Ad}_g(v)) = \mathrm{Ad}_{\rho(g)}(d\rho_e(v)).$$

We thus get the implication that *a homomorphism ρ respects the adjoint action of a group G on on its tangent space T_eG at the identity*.

While the above result is useful, it still involves the use of the homomorphism ρ. We rectify this by taking the derivative of Ad at the identity, noting that since $\mathrm{Aut}(T_eG) \subset \mathrm{End}(T_eG)$ (the endomorphisms of T_eG, that is, linear maps $T_eG \to T_eG$) is open there is a natural identification of $T_e\mathrm{Aut}(T_eG)$ with $\mathrm{End}(T_eG)$. This gives us a map

$$d(\mathrm{Ad})_e = \mathrm{ad} : T_eG \to \mathrm{End}(T_eG).$$

We can view ad as a map taking 2 inputs in $X, Y \in T_eG$ and giving one output $\mathrm{ad}_X(Y) \in T_eG$. In particular, under this identification ad becomes a bilinear map

$$T_eG \times T_eG \to T_eG.$$

We now define

$$[X, Y] = \mathrm{ad}_X(Y).$$

We now note that the following diagram commutes:

$$\begin{array}{ccc} T_eG & \xrightarrow{d\rho_e} & T_eH \\ \Big\downarrow{\scriptstyle \mathrm{ad}_X} & & \Big\downarrow{\scriptstyle \mathrm{ad}_{d\rho_e(X)}} \\ T_eG & \xrightarrow[d\rho_e]{} & T_eH \end{array}$$

that is,

$$dp_e([X, Y]) = [dp_e(X), dp_e(Y)] \text{ for all } X, Y \in T_eG.$$

We get the characterization that *the differential $d\rho_e$ of a homomorphism ρ on a Lie group G respects the adjoint action of the tangent space to G on itself*.

While the bracket operation $[-,-] : T_eG \times T_eG \to T_eG$ seems fairly abstract, there are many cases in which it can be made fairly explicit. For instance, take $G = \mathrm{GL}_n\mathbb{R}$. For any general matrix $X \in \mathbb{R}^{n\times n}$ we can find a smooth arc $\gamma : [0, 1] \to G$ such that $\gamma(0) = I$ and $\gamma'(0) = X$ (we will elaborate on this next lecture when we define the exponential map). It follows then that $T_eG = \mathbb{R}^{n\times n}$ and

$$\mathrm{Ad}_g(X) = \frac{d}{dt}\Psi_g(\gamma(t))\bigg|_{t=0} = \frac{d}{dt}(g\gamma(t)g^{-1})\bigg|_{t=0} = gXg^{-1},$$

in particular the adjoint action of G on T_eG is simply conjugation by elements of G. Letting $X, Y \in T_eG$ and $\gamma : [0, 1] \to G$ be as before we can compute using the product rule that

$$\begin{aligned} [X, Y] = \mathrm{ad}_X(Y) = &\ \frac{d}{dt}\,\mathrm{Ad}_{\gamma(t)}(Y)\bigg|_{t=0} \\ = &\ \frac{d}{dt}(\gamma(t)Y\gamma(t)^{-1})\bigg|_{t=0} \\ = &\ \gamma'(0)Y\gamma(0)^{-1} + \gamma(0)Y(-\gamma(0)^{-1}\gamma'(0)\gamma(0)^{-1}) \\ = &\ XY - YX. \end{aligned}$$

Thus, the bracket operation on the tangent space to $\mathrm{GL}_n\mathbb{R}$ is simply the commutator (hence the notation).

Remark. We provide a quick proof of why $(\gamma(t)^{-1})' = -\gamma(t)^{-1}\gamma'(t)\gamma(t)^{-1}$. Note that for all $t \in [0, 1]$ we have $I = \gamma(t)\gamma(t)^{-1}$. Using the product rule, we can differentiate both sides to obtain

$$0 = \gamma'(t)\gamma(t)^{-1} + \gamma(t)(\gamma(t)^{-1})' \implies \gamma(t)(\gamma(t)^{-1})' = -\gamma'(t)\gamma(t)^{-1}.$$

We can now multiply both sides by $\gamma(t)^{-1}$ to obtain the desired identity $(\gamma(t)^{-1})' = -\gamma(t)^{-1}\gamma'(t)\gamma(t)^{-1}$.

In the next lecture we will prove for a general Lie group G that given any $X \in T_eG$ there exists a homomorphism $\rho : \mathbb{R} \to G$ satisfying $\rho(0) = e$ and $\rho'(0) = X$ (the so-called one parameter subgroups of G). We will use the existence of such homomorphisms to prove the following Theorem:

Theorem. *Let G be a Lie group and let $\mathfrak{g} = T_eG$. Then the adjoint action $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfies the following properties:*

1. *Skew-commutatitivity: $[X, Y] = -[Y, X]$ for any $X, Y \in \mathfrak{g}$;*
2. *The Jacobi identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for any $X, Y, Z \in \mathfrak{g}$.*

Proof. We begin by proving (1). This is equivalent to the assertion that $[X, X] = 0$ for all $X \in \mathfrak{g}$, as assuming said assertion for any $X, Y \in \mathfrak{g}$ we can deduce

$$\begin{aligned} 0 &= [X + Y, X + Y] \\ &= [X, X] + [X, Y] + [Y, X] + [Y, Y] \\ &= [X, Y] + [Y, X] \\ &\implies [X, Y] = -[Y, X]. \end{aligned}$$

We thus proceed to show $[X, X] = 0$ for all $X \in \mathfrak{g}$. Let $\rho : \mathbb{R} \to G$ satisfying $\rho(0) = e$ and $\rho'(0) = X$. We can then compute

$$\begin{aligned} [X, X] &= \frac{d}{dt} \operatorname{Ad}_{\rho(t)}(X)\Big|_{t=0} = \frac{d}{dt}\left(\frac{d}{ds}\Psi_{\rho(t)}(\rho(s))\Big|_{s=0}\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\frac{d}{ds}\rho(t)\rho(s)\rho(t)^{-1}\Big|_{s=0}\right)\Big|_{t=0} = \frac{d}{dt}\left(\frac{d}{ds}\rho(t)\rho(s)\rho(-t)\Big|_{s=0}\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\frac{d}{ds}\rho(t + s - t)\Big|_{s=0}\right)\Big|_{t=0} = \frac{d}{dt}\left(\frac{d}{ds}\rho(s)\Big|_{s=0}\right)\Big|_{t=0} = \frac{d}{dt}X\Big|_{t=0} = 0. \end{aligned}$$

This completes the proof of (1).

In order to prove (2), we let $\rho : \mathbb{R} \to G$ be a homomorphism satisfying $\rho(0) = e$ and $\rho'(0) = Z$. We then can compute

$$\begin{aligned} [Z, [X, Y]] &= \frac{d}{dt} \operatorname{Ad}_{\rho(t)}([X, Y])\Big|_{t=0} = \frac{d}{dt} d(\Psi_{\rho(t)})_e([X, Y])\Big|_{t=0} \\ &= \frac{d}{dt}\left[d(\Psi_{\rho(t)})_e(X), d(\Psi_{\rho(t)})_e(Y)\right]\Big|_{t=0} = \frac{d}{dt}\left[\operatorname{Ad}_{\rho(t)}(X), \operatorname{Ad}_{\rho(t)}(Y)\right]\Big|_{t=0} \\ &= \left[\frac{d}{dt}\operatorname{Ad}_{\rho(t)}(X)\Big|_{t=0}, Y\right] + \left[X, \frac{d}{dt}\operatorname{Ad}_{\rho(t)}(Y)\Big|_{t=0}\right] \\ &= [[Z, X], Y] + [X, [Z, Y]]. \end{aligned}$$

Using (1) we can rearrange the above to obtain $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, proving (2). ■

Using the above theorem as inspiration we are led to the following definition:

Definition (Lie algebra)**.** A Lie algebra $\mathfrak{g}$ is a vector space along with a skew-symmetric bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity. The bilinear map is commonly referred to as a Lie bracket on $\mathfrak{g}$. Given two Lie algebras $\mathfrak{g}, \mathfrak{h}$ a linear map $T : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism if it respects the Lie brackets, that is if

$$T([X, Y]_{\mathfrak{g}}) = [T(X), T(Y)]_{\mathfrak{h}} \quad \text{for all } X, Y \in \mathfrak{g}.$$

If G is a Lie group then $\mathfrak{g} = T_eG$ becomes a Lie algebra with Lie bracket given by the adjoint action of the tangent space on itself. In this case $\mathfrak{g}$ is known as the Lie algebra of G.

Examples of Lie algebras

Before giving examples of Lie algebras we make a remark about notation. From this point forward the Lie algebra of a Lie group G will be denoted by the fraktur character $\mathfrak{g}$. For instance the Lie algebra of $\mathrm{GL}_n\mathbb{R}$ is commonly denoted as $\mathfrak{gl}_n\mathbb{R}$ and even more generally the Lie algebra of $\mathrm{GL}(V)$ is denoted as $\mathfrak{gl}(V)$. We showed above that as a vector space $\mathfrak{gl}_n\mathbb{R} \cong \mathbb{R}^{n\times n}$ (and more generally $\mathfrak{gl}(V) \cong \mathrm{End}(V)$), however when we refer to a Lie algebra $\mathfrak{g}$ for a known Lie group G we are always referring to the underlying vector space *along* with its inherited Lie bracket.

The first Lie group we will consider is $\mathrm{SL}_n\mathbb{R}$. Recall that for all $X \in \mathrm{SL}_n\mathbb{R}$ we have $\det(X) = 1$. We will use the following Lemma to find $\mathfrak{sl}_n\mathbb{R}$.

Lemma. *Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ be differentiable and $\gamma(0) = I$. Then*

$$\frac{d}{dt}\det(\gamma(t))\bigg|_{t=0} = \mathit{Tr}\,\gamma'(0).$$

Proof. Let $\gamma_{ij}(t)$ denote the (i,j) entry of $\gamma(t)$ and let $\gamma_{\hat{i}\hat{j}}(t)$ denote $\gamma(t)$ without the i^{th} row and j^{th} column. We can then write that

$$\det(\gamma(t)) = \sum_{j=1}^{n}(-1)^{j+1}\gamma_{1j}(t)\det(\gamma_{\hat{1}\hat{j}}(t)).$$

Taking the derivative of both sides at zero we obtain

$$\begin{aligned}\frac{d}{dt}\det(\gamma(t))\bigg|_{t=0} &= \frac{d}{dt}\left[\sum_{j=1}^{n}(-1)^{j+1}\gamma_{1j}(t)\det(\gamma_{\hat{1}\hat{j}}(t))\right]\Bigg|_{t=0}\\ &= \sum_{j=1}^{n}\left[(-1)^{j+1}\gamma'_{1j}(t)\det(\gamma_{\hat{1}\hat{j}}(0)) + (-1)^{j+1}\gamma_{1j}(0)\,\frac{d}{dt}\det(\gamma_{\hat{1}\hat{j}}(t))\bigg|_{t=0}\right]\\ &= \gamma'_{11}(0) + \frac{d}{dt}\det(\gamma_{\hat{1}\hat{1}}(t))\bigg|_{t=0}\end{aligned}$$

where the final equality uses the fact $\gamma(0) = I$. Repeating recursively we see that

$$\frac{d}{dt}\det(\gamma(t))\bigg|_{t=0} = \gamma'_{11}(0) + \gamma'_{22}(0) + \cdots + \gamma'_{nn}(0) = \mathrm{Tr}\,\gamma'(0)$$

as claimed. ■

Now assume $X \in \mathfrak{sl}_n\mathbb{R}$ and let $\rho_X : \mathbb{R} \to \mathrm{SL}_n\mathbb{R}$ be the corresponding homomorphism with $\rho_X(0) = I$ and $\rho'_X(0) = X$. Since $\det(\rho(t)) = 1$ for all $t \in \mathbb{R}$ the above Lemma implies $\mathrm{Tr}\,X = 0$. We see that the matrices in $\mathbb{R}^{n\times n}$ with trace 0 form an n^2-1 dimensional of $\mathfrak{sl}_n\mathbb{R}$, and since $\mathrm{SL}_n\mathbb{R}$ is n^2-1 dimensional, $\mathfrak{sl}_n\mathbb{R} = \{X \in \mathfrak{gl}_n\mathbb{R} : \mathrm{Tr}\,X = 0\}$ with Lie bracket being the commutator. We note that in the above analysis we could of chosen $\mathbb{C}$ instead and none of the computation would of changed.

We now consider the Lie groups defined by preservation of a bilinear form Q. If G is such a Lie group, $X \in \mathfrak{g}$ and let $\rho_X : \mathbb{R} \to G$ the corresponding homomorphism with $\rho_X(0) = I$ and $\rho'_X(0) = X$ we then have that

$$Q(\rho(t)v, \rho(t)w) = Q(v, w).$$

Deriving both sides with respect to t at 0 we obtain

$$Q(Xv, w) + Q(v, Xw) = 0.$$

If Q has a matrix representation M this is equivalent to the condition

$$X^T M + MX = 0.$$

If $G = \mathrm{O}(n)$ or $G = \mathrm{SO}_n\mathbb{R}$ the standard inner product on $\mathbb{R}^n$ has matrix representation I, so by comparing dimensions we see

$$\mathfrak{o}(n) = \mathfrak{so}_n\mathbb{R} = \left\{X \in \mathbb{R}^{n\times n} : X^T = -X\right\},$$

that is both $\mathfrak{o}(n)$ and $\mathfrak{so}_n\mathbb{R}$ are the skew symmetric $n \times n$ matrices. If $G = \mathrm{Sp}_{2n}\mathbb{R}$ we take M to be the matrix

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

representing the standard skew-symmetric, nondegenerate bilinear form on $\mathbb{R}^{2n}$. In this case comparing dimensions gives

$$\mathfrak{sp}_{2n}\mathbb{R} = \left\{X \in \mathbb{R}^{n\times n} : X^T\Omega + \Omega X = 0\right\}.$$

Left-invariant vector fields and the exponential map

We now finish the lecture by discussing something seemingly unrelated to everything previously which we use to show the existence of the homomorphisms $\rho : \mathbb{R} \to G$ satisfying $\rho(0) = e$ and $\rho'(0) = X$ for any $X \in \mathfrak{g}$. Let $v : M \to TM$ be a vector field, i.e. a smooth map into the tangent bundle such that if $\pi : TM \to M$ is the projection map we have $\pi \circ v = \mathrm{Id}_M$ (that is, v is a section of TM). Now assume that some Lie group G has a smooth left action on M. Then for each $g \in G$ we obtain a diffeomorphism $L_g : M \to M$ given by $m \mapsto gm$. We say that v is a left-invariant vector field if $d(L_g)_e(v(m)) = v(gm)$ for all $g \in G$ and $m \in M$.

In particular consider a Lie group G acting on itself via left multiplication. We can construct a left-invariant vector field on G simply by choosing some $X \in \mathfrak{g}$ and setting $v_X(g) = d(L_g)_e(X)$. It is simply to see that conversely, any left-invariant vector field on G by this action is uniquely determined by its value $v(e) \in \mathfrak{g}$. It turns out that that the bracket of two left-invariant vector fields is once again left-invariant (where the bracket is the standard bracket on vector fields given by the "commutator") and satisfies the axioms of a Lie algebra. In fact, while we won't prove it, the Lie algebra $\mathfrak{g}$ obtained by differentiating the conjugation automorphism and the Lie algebra obtained from left-invariant vector fields on G under the action of left multiplication are isomorphic.

Now choose $X \in \mathfrak{g}$ and take the left-invariant vector field $v_X(g) = d(L_g)_e(X)$. A basic theorem from differential equations allows us to find some integral curve $\rho : (-\varepsilon, \varepsilon) \to G$ with $\rho(0) = e$ for small $\varepsilon > 0$ such that ρ is uniquely determined by the condition

$$\rho'(t) = v_X(\rho(t)).$$

We claim this ρ is a homomorphism on this interval, that is $\rho(s + t) = \rho(s)\rho(t)$. Let $\alpha(t) = \rho(s + t)$ and $\beta(t) = \rho(s)\rho(t)$. It is easy to check $\alpha'(t) = v_X(\alpha(t))$ and $\beta'(t) = v_X(\beta(t))$ using the left-invariance of v_X. It follows then that $\alpha(t)$ and $\beta(t)$ are equal since $\alpha(0) = \beta(0)$ and integral curves are unique. We can now extend ρ to all of $\mathbb{R}$ by defining $\rho(s + t) = \rho(s)\rho(t)$ if ρ is defined at $s, t \in \mathbb{R}$ and not $s + t \in \mathbb{R}$.

The ρ we have constructed are the unique homomorphisms asserted to exist previously. Commonly we refer to such ρ as one-parameter subgroups of G and for $X \in \mathfrak{g}$ we denote

$$\rho(t) = \exp(tX) = e^{tX}.$$

Fixing $t = 1$ we obtain a map $\exp : \mathfrak{g} \to G$ defined by $X \mapsto e^X$. This map is referred to as the exponential map and gives us a gadget for taking elements of a Lie algebra and recovering elements of a Lie group. We will go into more detail on $\exp$ in the next lecture.

Lecture 2

Representations and the Baker-Campbell-Hausdorff Formula

Representation Theory

Representation theory is a very powerful tool in mathematics, allowing for difficult questions that arise in studying abstract algebra to be transferred to the much more understood subject of Linear algebra. In fact, representation theory finds many uses not only in abstract algebra, but also analysis, geometry, number theory, and physics. Now that we have defined a Lie group and a Lie algebra we are ready to dive into the meat of the lecture and discuss their corresponding representation theory. We begin with general definitions pertaining to group representations.

> **Definition** (Group Representation)**.** Let K be a field. An K-representation of a group G is a pair (π, V) where V is a vector space over K and $\pi : G \to \mathrm{GL}(V)$ is a homomorphism of groups. The dimension of a representation is equal to the dimension of V as a vector space.

Note that a representation of G defines a G-module structure on V. Thus, when the underlying representation (π, V) is understood, we will use the shorthand $\pi(g)v = gv$ for $g \in G$ and $v \in V$. Since a representation of G gives V a G-module structure we can define a homomorphism between two representations to be a G-linear map on V in the following sense:

> **Definition** (Morphism of Group Representations)**.** Let G be a group and (π_1, V), (π_2, W) be two K-representations of G. A homomorphism between (π_1, V) and (π_2, W) is a linear transformation $T : V \to W$ such that $T(\pi_1(g)v) = \pi_2(g)T(v)$ for all $g \in G$ and $v \in V$.

Maps as described above are often called G-equivariant maps or intertwining maps. Two K-representations of a group G are said to be isomorphic if there exists an invertible intertwining map between them.

We now discuss the above in the context of Lie groups. First, since G admits a smooth structure we are mainly interested in fields K such that for a vector space V over K the group $\mathrm{GL}(V)$ also admits a smooth structure. Thus when speaking about Lie groups the natural choices for K are the field of real numbers $\mathbb{R}$ and the field of complex numbers $\mathbb{C}$. Second, given $\mathrm{GL}(V)$ is a Lie group we require our representations (π, V) be smooth (i.e. π is a Lie group homomorphism). Given these two conditions we can differentiate π to obtain a Lie algebra map $d\pi_e : \mathfrak{g} \to \mathfrak{gl}(V)$. This leads us to define the following:

> **Definition** (Lie Algebra Representation)**.** A representation of a real (complex) Lie algebra $\mathfrak{g}$ is a pair (α, V) where V is a real (complex)vector space and $\alpha : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra map. The dimension of a representation is equal to the dimension of V as a vector space. (Recall $\mathfrak{gl}(V) = \mathrm{End}(V)$ with the commutator bracket).

Intertwining maps for Lie algebra representations are defined analogously to that of groups.

Definition (Morphism of Lie algebra Representations)**.** Let $\mathfrak{g}$ be a Lie algebra and (α_1, V), (α_2, W) be two representations of $\mathfrak{g}$. A homomorphism between (α_1, V) and (α_2, W) is a linear transformation $T : V \to W$ such that $T(\alpha_1(X)v) = \alpha_2(X)T(v)$ for all $X \in \mathfrak{g}$ and $v \in V$.

Also analogously to groups, we say two representations of $\mathfrak{g}$ are isomorphic if there exists an invertible intertwining map between them.

We state a couple facts about the above discussion. First, the Jacobi Identity is equivalent for Lie algebras arising from Lie groups to the statement that $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation. We will thus often refer to this map as the adjoint representation of a Lie algebra. Second, a representation is called faithful if it is injective. We state, but not prove, the following fundemental result about faithful representations of Lie algebras:

Theorem (Ado's Theorem)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the real (or complex) numbers. Then there exists some finite dimensional real (or complex) vector space V such that $\mathfrak{g}$ has a faithful representation into V.*

Ado's theorem lets us regard any finite dimensional Lie algebra over the real (or complex) numbers as some set of square matrices over a vector space with bracket the commutator. We will use this fact repeatedly throughout this lecture.

The last thing we discuss is invariant subspace of a representation. Let (π, V) be a representation of a group G. We say that subspace $W \subset V$ is invariant with respect to (π, V) if $gW \subset W$ for all $g \in G$. If (π, V) has no proper invariant subspaces we call the representation irreducible. We remark that the same definition of invariant subspace carries over to representations of a Lie algebra $\mathfrak{g}$, and in fact, if $W \subset V$ is invariant under a representation (π, V) of G then it is also invariant under the representation $(d\pi_e, V)$ of $\mathfrak{g}$. Thus, a representation (π, V) is irreducible if and only if $(d\pi_e, V)$ is irreducible.

Isogeny

We depart for a moment from the previous discussion to talk about some general topology we will need to continue our study. Let X be a topological space. We will first define what we mean when we say X is a simply connected space.

Definition (Simply Connected)**.** A space X is said to be *simply connected* if and only if it is path-connected and any loop in X defined by $f : S^1 \to X$ can be contracted to a point, that is, there is a map $F : D^2 \to X$ such that $F|_{\partial D^2} = f$. Equivalently, X is path-connected and trivial fundamental group at each point.

We also define what we mean by a covering space of X.

Definition (Covering Space)**.** A covering space of a topological space X is a topological space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying the following condition: each point $x \in X$ has an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open subsets in $\widetilde{X}$ each of which is mapped homeomorphically onto U by p. In the case $\widetilde{X}$ is simply connected it is called the universal cover of X and is unique up to homeomorphism respecting the covering space maps.

Let $p : \widetilde{X} \to X$ be a covering space and $f : Y \to X$. We say that $\widetilde{f} : Y \to \widetilde{X}$ is a lift of f if $\widetilde{f} \circ p = f$. It turns out we can completely characterize when a map $f : Y \to X$ will lift to a map $\widetilde{f} : Y \to \widetilde{X}$, see [Hat02] for details. This lifting property lets us deduce the following proposition:

Proposition. *Let G be a Lie group, H a connected manifold, and $p : H \to G$ a covering space map. Let e' be a lift of the identity e in G (i.e. $p(e') = e$). Then there is a unique Lie group structure on H such that e' is the identity and p is a map of Lie groups; and the kernel of p is in the center of H.*

The above is deduced by lifting the multiplication map $G \times G \to G$ to the universal cover $\widetilde{G}$ and using the following "converse" proposition for the intermediate covers:

Proposition. *Let H be a Lie group, and $\Gamma \subset Z(H)$ a discrete subgroup of its center. Then there is a unique Lie group structure on the quotient group $G = H/\Gamma$ such that the quotient map $H \to G$ is a Lie group map.*

We now define an equivalence relation on Lie groups. We call a Lie group map $G \to H$ an isogeny if it is a covering space map and call G and H isogenous if there is an isogeny between them. Isogeny is not quite an equivalence relation, however we take the equivalence relation it generates. We note two key features of the isogeny equivalence relation. First, every isogeny equivalence class of a Lie group G has an initial member given by the universal cover $\widetilde{G}$. Second, two isogenous Lie groups have isomorphic Lie algebras. We will see later that every finite dimensional Lie algebra can be associated to a unique isogeny class of Lie groups.

Returning to the Exponential Map

Last time we constructed for any arbitrary Lie group G and its Lie algebra $\mathfrak{g}$ the unique homomorphisms $\rho_X : \mathbb{R} \to G$ (one-parameter subgroups) satisfying $\rho_X(0) = e$ and $\rho'_X(0) = X$ for each $X \in \mathfrak{g}$ and defined $\rho_X(t) := e^{tX}$ and $\exp : \mathfrak{g} \to G$ as the map $\exp(X) = e^X$. The map $\exp$ is called the exponential of the Lie algebra. We note in general $\exp$ is not surjective, however given our Lie group is connected this does not matter as a result of the following theorem:

Theorem. *The exponential map is the unique map from $\mathfrak{g}$ to G taking 0 to e whose differential at the origin $d\exp_0 : T_0\mathfrak{g} = \mathfrak{g} \to T_eG = \mathfrak{g}$ is the identity, and whose restrictions to the lines through the origin in $\mathfrak{g}$ are one-parameter subgroups of G.*

The theorem follows essentially from the fact that any line through the origin of $\mathfrak{g}$ can be written as tX for some $X \in \mathfrak{g}$ and the fact $\rho_{tX}(s) = \rho_X(ts)$ by uniqueness of one-parameter subgroups. In particular, we can deduce from this that given a homomorphism $\Psi : G \to H$ the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\psi_e} & \mathfrak{h} \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\psi]{} & H \end{array}$$

commutes. Since $d\exp_0$ is an isomorphism, it follows from the inverse function theorem that $\exp$ is a local diffeomorphism at the origin of $\mathfrak{g}$ and thus maps a sufficiently small open disc $\Delta \subset \mathfrak{g}$ centered at the origin of $\mathfrak{g}$ diffeomorphically onto a open set $\exp(\Delta) \subset G$. If G is connected then $\exp(\Delta)$ generates all of G by the theorem proven last time, and thus according to the above diagram, ψ is completely determined by $d\psi_e$.

Now let $X \in \mathfrak{gl}(V)$ where V is a real (or complex) vector space. Recall from elementary calculus that for real numbers we can represent the exponential e^t by its taylor series, i.e.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!},$$

for all values of t. We now define $X^n = \prod_{i=1}^n X$ and define

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

It turns out this map converges for all $X \in \mathfrak{gl}_n(V)$ and is a map into $\mathrm{GL}(V)$. The map defined above is referred to as the matrix exponential. It is simple computation to show that $e^{sX}e^{tX} = e^{(s+t)X}$, $e^0 = I$, and $\frac{d}{dt}e^{tX}\Big|_{t=0} = X$. We can therefore conclude that e^{tX} is precisely the one-parameter subgroup corresponding to X. Therefore we deduce if $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$, the exponential map is precisely the matrix exponential. We will use this fact in the next section to show that the group law on G is locally determined by the Lie algebra structure on $\mathfrak{g}$.

The Baker-Campbell-Hausdorff Formula

We now know that a Lie group map is completely determined by its differential at the identity, but now we would like to classify which Lie algebra maps arise as differentials of a Lie group map, the end goal being the construction of a Lie group representation from a Lie algebra representation. We note currently that we currently only have one tool linking a Lie group G to its Lie algebra $\mathfrak{g}$, the exponential map $\exp : \mathfrak{g} \to G$. Our first goal then should be an attempt to recover the group operation of G simply from the Lie algebra structure of $\mathfrak{g}$. That is, we want to define some partial function (maybe total) $* : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$e^{X*Y} = e^X e^Y.$$

To begin, we will use Ado's Theorem to view $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{gl}(V)$. We can then view the exponential map as the matrix exponential so that

$$\begin{aligned} e^{X*Y} &= e^X e^Y \\ &= \left(I + X + \frac{X^2}{2} + \cdots\right)\left(I + Y + \frac{Y^2}{2} + \cdots\right) \\ &= I + (X+Y) + \left(\frac{X^2}{2} + XY + \frac{Y^2}{2}\right) + \cdots \end{aligned}$$

If $G \subset \mathrm{GL}(V)$ we can define for $g \in G$ sufficiently close to the identity $I \in G$ a partial inverse to the exponential map given by

$$\log(g) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(g-I)^n}{n}.$$

We can use this to compute

$$X * Y = \log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\big([X,[X,Y]] + [Y,[Y,X]]\big) - \cdots .$$

This formula is known as the Baker-Campbell-Hausdorff formula and can be completely expressed in terms of the Lie bracket. The Baker-Campbell-Hausdorff formula may not converge for all $X, Y \in \mathfrak{g}$ but will for X and Y suitably small (i.e. small enough that $\log(e^X e^Y)$ is defined).

While we have not written out a closed form for the Baker-Campbell-Hausdorff formula, the existence of the formula is more important than actually using it in computation. The first theorem we can prove with this formula is the following:

Theorem. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then the subgroup of the group G generated by $\exp \mathfrak{h}$ is an immersed subgroup H with tangent space $T_e H = \mathfrak{h}$.*

The basic idea is that we need only look at the exponential of small open discs $\Delta \subset \mathfrak{h}$ about the origin and prove that $\exp(\Delta) \cdot \exp(\Delta) \subset \exp \mathfrak{h}$ (since $\exp(\Delta)$ and $\exp \mathfrak{h}$ will generate the same subgroup). We now use the fact that G is isogenous to some subgroup of $\mathrm{GL}(V)$ (via Ado's Theorem) and the Baker-Campbell-Hausdorff formula to deduce the theorem.

The next two theorems essentially will prove that the category of Lie algebras is equivalent to the full subcategory of Lie groups consisting only of simply connected Lie groups. The first result is known as Lie's second theorem:

Theorem (Lie's Second Theorem)**.** *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, such that G is simply connected. If $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra map there is a unique map $F : G \to H$ of Lie groups such that $dF_e = f$.*

Proof. The idea of this proof is to consider the product $G \times H$ with Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$ and notice that f being a map of Lie algebras is equivalent to the graph $\mathfrak{j} \subset \mathfrak{g} \oplus \mathfrak{h}$ of f being a Lie subalgebra. Using the previous theorem we associate an immersed subgroup $J \subset G \times H$ to $\mathfrak{j}$. Now projection $\pi : J \to G$ induces an isomorphism between $\mathfrak{j}$ and $\mathfrak{g}$, and thus, π is an isogeny (via Lie's Third Theorem to be stated). Since G is simply connected π is an isomorphism so $J \cong G$. Finally, the projection $\eta : G \to H$ onto the second factor is the desired map. ■

The next result is known as Lie's Third theorem:

Theorem (Lie's Third Theorem)**.** *If $\mathfrak{g}$ is a finite dimensional Lie algebra there exists a unique simply connected Lie group G with Lie algebra $\mathfrak{g}$.*

Proof. Use Ado's Theorem to embed $\mathfrak{g}$ into $\mathfrak{gl}(V)$. Then there exists an immersed Lie subgroup $G_0 \subset \mathrm{GL}(V)$ with Lie algebra $\mathfrak{g}$ (by a previous theorem). Take the univeral cover G of G_0. G is simply connected and has Lie algebra $\mathfrak{g}$. Uniqueness follows from the fact that the isomorphism class of a simply connected Lie group is determined by its Lie algebra. ■

Using the above two theorems we can establish the following principle:

> Representations of a simply connected Lie groups are in bijective correspondence with representations of finite dimensional Lie algebras.

In order to study the representation theory of Lie groups we therefore need only to study the representation theory of Lie algebras.

Representations of Lie Algebras

With the justification out of the way we are now ready to study the representation theory of Lie algebras. We begin with a long list of definitions that will come in handy later.

Definitions. Let $\mathfrak{g}$ be a Lie algebra. We make the following defintions:

(I) the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is the subspace of $\mathfrak{g}$ consisting of elements $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$.

(II) $\mathfrak{g}$ is abelian if $Z(\mathfrak{g}) = \mathfrak{g}$, i.e. if all Lie brackets are zero.

(III) a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

(IV) the lower central series $\mathcal{D}_k\mathfrak{g}$ is defined by $\mathcal{D}_1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathcal{D}_k = [\mathcal{D}_{k-1}\mathfrak{g}, \mathfrak{g}]$. (Note each $\mathcal{D}_k\mathfrak{g}$ is an ideal of $\mathfrak{g}$)

(V) the derived series $\mathcal{D}^k\mathfrak{g}$ is defined by $\mathcal{D}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathcal{D}^k = \left[\mathcal{D}^{k-1}\mathfrak{g}, \mathcal{D}^{k-1}\mathfrak{g}\right]$. (Note each $\mathcal{D}^k\mathfrak{g}$ is an ideal of $\mathfrak{g}$)

(VI) $\mathfrak{g}$ is nilpotent if $\mathcal{D}_k\mathfrak{g} = 0$ for some k.

(VII) $\mathfrak{g}$ is solvable if $\mathcal{D}^k\mathfrak{g} = 0$ for some k.

(VIII) $\mathfrak{g}$ is semisimple if $\mathfrak{g}$ has no nonzero solvable ideals.

(IX) $\mathfrak{g}$ is simple if $\dim(\mathfrak{g}) > 1$ and it contains no nontrivial ideals.

We commonly refer to $\mathcal{D}^1 = \mathcal{D}_1$ as the commutator subalgebra of $\mathfrak{g}$ and denote it via $\mathcal{D}\mathfrak{g}$. We also note that $\mathfrak{h} \subset \mathfrak{g}$ induces a Lie bracket on the quotient vector space $\mathfrak{g}/\mathfrak{h}$ if and only if $\mathfrak{h}$ is an ideal. One thing we could prove is that $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable for ideals $\mathfrak{h} \subset \mathfrak{g}$. In particular, this allows us to deduce that the sum of any two solvable ideals in a Lie algebra are once again solvable. We define the radical of $\mathfrak{g}$, Rad($\mathfrak{g}$), to be the sum of all solvable ideals in $\mathfrak{g}$. Rad($\mathfrak{g}$) then becomes a maximal solvable ideal in $\mathfrak{g}$. The quotient $\mathfrak{g}/\text{Rad}(\mathfrak{g})$ is semisimple, and we obtain an exact sequence

$$0 \to \text{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\text{Rad}(\mathfrak{g}) \to 0$$

With this as preliminary justification, in order to study the representation theory of an arbitrary Lie algebra $\mathfrak{g}$ we essentially need to study the representation theory of solvable and semisimple Lie algebras.

We begin with the case $\mathfrak{g}$ is solvable. The representation theory of $\mathfrak{g}$ starts with a proof of Engel's theorem:

Theorem (Engel's Theorem)**.** *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be any Lie subalgebra such that every $X \in \mathfrak{g}$ is a nilpotent endomorphism of V. Then there exists a nonzero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$.*

From here we deduce Lie's theorem:

Theorem (Lie's Theorem)**.** *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a complex solvable Lie algebra. Then there exists a nonzero vector $v \in V$ that is an eigenvector of X for all $X \in \mathfrak{g}$.*

Lie's theorem now lets us deduce the irreducible representations of a complex solvable Lie algebra are all one dimensional. In fact, we can deduce even more:

Theorem. *Let $\mathfrak{g}$ be a complex Lie algebra, $g_{ss} = \mathfrak{g}/Rad(\mathfrak{g})$. Every irreducible representation of $\mathfrak{g}$ is of the form $V = V_0 \otimes L$, where V_0 is an irreducible representation of $\mathfrak{g}_{ss}$ and L is a one-dimensional representation.*

We now restrict our attention for the moment to semisimple Lie algebras. A key fact we will make use of is:

Theorem (Complete Reducibility)**.** *Let V be a representation of the semisimple Lie algebra $\mathfrak{g}$ and $W \subset V$ a subspace invariant under the action of $\mathfrak{g}$. Then there exists a subspace $W' \subset V$ complementary to W and invariant under $\mathfrak{g}$.*

This allows us to decompose any representation of a semisimple Lie algebra into a direct sum of irreducible representations. Thus, in order to study the representation theory of a semisimple Lie algebra we need only find it's irreducible representations.

Lecture 3

Representation Theory of $\mathfrak{sl}_2\mathbb{C}$

Preliminaries

Before we analyze the representations of $\mathfrak{sl}_2\mathbb{C}$ in detail we'll need a fact about the behavior of algebraic properties under a representation of Lie algebras, the preservation of Jordan Decomposition:

Definition. Let V be a vector space and $X \in \text{End}(V)$ an operator. Then X is *semisimple* if every X-invariant subspace has an X-invariant complement. Over an algebraically close field, an operator is semisimple iff it is diagonalizable .

Theorem. *Let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. Then for any $X \in \mathfrak{g}$ we have $X = X_{ss} + X_n$, where X_{ss} is semisimple, X_n is nilpotent, and both X_{ss} and X_n are in $\mathfrak{g}$. Furthermore, for any representation ρ, the decomposition $\rho(X) = \rho(X_{ss}) + \rho(X_n)$ is a decomposition of $\rho(X)$ into semisimple and nilpotent parts.*

The coincidence of semisimplicity and diagonalizablity motivate us to consider Lie algebras over $\mathbb{C}$ instead of $\mathbb{R}$. Another motivation is the phenomenon of *complexification*. Essentially, we can have non-isomorphic real Lie algebras which have isomorphic tensor products with $\mathbb{C}$, which complicates the picture.

Lastly, we have the following theorem, which is involved but not difficult.

Theorem. *With five exceptions, every simple complex Lie algebra is isomorphic to either $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{so}_n\mathbb{C}$, or $\mathfrak{sp}_{2n}\mathbb{C}$ for some n.*

With this theorem in mind, we turn to the representations of $\mathfrak{sl}_2\mathbb{C}$, pausing to recall that $\mathfrak{sl}_2\mathbb{C}$ is semisimple.

Overview

Fully understanding the representation theory of $\mathfrak{sl}_2\mathbb{C}$ is important for a variety of reasons:

- The representation theory of $\mathfrak{sl}_n\mathbb{C}$ is, overall, very similar. The modifications are mostly technical, not conceptual.
- The analysis of arbitrary semisimple Lie algebras over $\mathbb{C}$ makes nontrivial use of sub-algebras isomorphic to $\mathfrak{sl}_2\mathbb{C}$.

Our goal will be to find and classify all the irreducible representations of $\mathfrak{sl}_2\mathbb{C}$. As a reminder, this is the set of all traceless 2×2 matrices, and is the Lie algebra of $\text{SL}_2\mathbb{C}$.

Notation

We have the following basis for $\mathfrak{sl}_2\mathbb{C}$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The commutation relations are

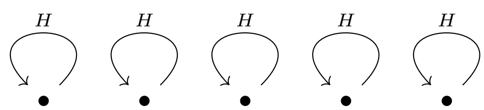
$$[H, R] = 2R, \quad [H, L] = -2L, \quad [R, L] = H.$$

The Lie algebra of matrix group has a maximal simultaneously diagonalizable subalgebra $\mathfrak{h}$. In this case $\mathfrak{h}$ is just the span of H but for other algebras (like $\mathfrak{sl}_3\mathbb{C}$), $\mathfrak{h}$ will have more structure. Still, $\mathfrak{h}$ controls a lot of the picture.

R and L can be thought of as raising and lowering operators if you're from physics, but "right" and "left" make fine names as well.

Analysis

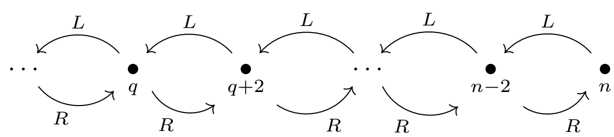
Let V be an irreducible representation of $\mathfrak{g}$. The preservation of Jordan decomposition implies that the action of $\mathfrak{h} = \text{span}\{H\}$ is diagonalizable, so we decompose V into eigenspaces of H, like so. Each dot is a different eigenspace.



Since V is finite dimensional, there exists a maximal eigenvalue of H, which we call n. We will see soon that these are integers. But first we use the commutator relations to involve L and R in the picture: Let v be an eigenvector of H with eigenvalue λ. Then we compute:

$$\begin{aligned} HLv &= LHv + [H, L]v \\ &= \lambda Lv - 2Lv \\ &= (\lambda - 2)Lv \end{aligned}$$

This trick is very useful. Repeating the same computation for R, we see that if v is an eigenvector of H with eigenvalue λ, then Rv is an eigenvector with eigenvalue $\lambda + 2$. So now our picture looks something like this, where we label eigenspaces with their eigenvalue and omit the H arrows.



We will now denote the eigenspace of H with eigenvalue λ by V_λ. By the irreducibility of V, we know that all the eigenvalues that occur for H are congruent mod 2, since otherwise the equivalence classes of eigenvalues (mod 2) give nontrivial invariant subspaces. Also, $R(V_n) \subset V_{n+2} = 0$. This picture *does* cheat somewhat, since we currently don't know that starting with a "highest weight" vector (eigenvector of H with eigenvalue n) generates the representation.

Lemma. *Let $v \in V_n$ be nonzero (so that we must have $R(v) = 0$). Then the vectors $(v, L(v), L^2(v), \dots)$ span V.*

Proof. Here we use the irreducibility of V to move to the condition that the set of vectors generated by L is invariant. It's invariant under L and H trivially. The computation for R gives some insight, so we repeat it here. We proceed by induction to establish the formula

$$R(L^k(v)) = k(n - k + 1)L^{k-1}(v),$$

which is stronger than we need. $R(v) = 0$, so the base case is established. The formula follows since

$$\begin{aligned} R(L^k(v)) &= [R, L]L^{k-1}(v) + LRL^{k-1}v \\ &= HL^{k-1}(v) + (k-1)(n-k+2)L^{k-1}(v) \\ &= (n-2k+2)L^{k-1}(v) + (k-1)(n-k+2)L^{k-1}(v) \\ &= k(n-k+1)L^k(v) \end{aligned}$$

■

Remark. This lemma is true in higher generality. As we'll see in the next lecture, the generalization comes from replacing the notion of eigenvalue.

Lemma. *An irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is determined entirely by the largest eigenvalue of H.*

Proof. Think about the proof of the previous lemma. ■

Now we use the finite dimensionality again to produce a lowest eigenvalue of H, called m.

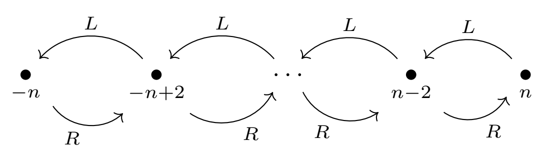
Lemma. *n is an integer, and $m = -n$. That is, the eigenvalues of H are symmetric about 0.*

Proof. We know $L^{k-1} \in V_m$ for some k so $L^k = 0$. But then

$$0 = RL^k(v) = k(n-k+1)L^{k-1}v$$

so that $n - k + 1 = 0$, or $n = k - 1$. We also have $m = (n - 2(k-1))$ since applying L lowers eigenvalues by 2. This gives $m = -n$. ■

So the picture for any irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ looks like



The last thing we'll want to do is to explicitly represent the representation with highest eigenvalue n. We need a bit more linear algebra for this first.

Definition. Let (V, ρ) and (V', ρ') be representations of a Lie algebra $\mathfrak{g}$. Then the *tensor product* of V and W' is the representation $(V \otimes V', \phi)$ with action

$$\phi(X)[v \otimes v'] = \rho(X)v \otimes v' + v \otimes \rho'(x)v'$$

Definition. Let V be a vector space. The *kth symmetric power* of V, $\mathrm{Sym}^k V$ is the quotient of the order k tensors by the ideal generated by all elements of the form

$$v \otimes v' - v' \otimes v$$

That is, we force the tensors to commute with each other. This has an easy interpretation as degree k polynomials in the elements of V, where the variables commute normally.

Returning to the analysis of $\mathfrak{sl}_2\mathbb{C}$, we first start with the standard $n = 2$ case. Let V be the free vector space on $\{x, y\}$. Then we have a representation of $\mathfrak{sl}_2\mathbb{C}$ given by the following relations:

Then we can extend this to $\mathrm{Sym}^k(V)$, which has basis $\{x^n, x^{n-1}y, \dots, xy^{n-1}, y^n\}$

When $n = 3$, this looks like (using the rule for tensor products to compute)

$$\begin{array}{ccc} H(x^2) = 2x^2 & H(xy) = 0 & H(y^2) = -2y^2 \\ R(x^2) = 0 & R(xy) = x^2 & R(y^2) = 2xy \\ L(x^2) = 2xy & L(xy) = y^2 & L(y^2) = 0 \end{array}$$

Remember, the structure of the entire representation follows from the first line. In general, we have $H(x^{n-k}y^k) = (n-2k)x^{n-k}y^k$, so this representation has highest eigenvalue n, and all the eigenspaces for H are one dimensional. This is enough to show that this representation is the irreducible representation with highest eigenvalue n.

The last topic is the decomposition of tensor products by diagrams, where we establish, basically by counting, the formula

$$\mathrm{Sym}^a V \otimes \mathrm{Sym}^b V = \mathrm{Sym}^{a+b} V \oplus \mathrm{Sym}^{a+b-2} V \oplus \cdots \mathrm{Sym}^{a-b} V$$

Lecture 4

Representation Theory of $\mathfrak{sl}_3\mathbb{C}$ and Beyond

Overview

Our goal now will be to highlight how the representation theory of a generic semisimple Lie algebra is carried out using $\mathfrak{sl}_3\mathbb{C}$ as our primary example. As mentioned previously, the representation theory of $\mathfrak{sl}_2\mathbb{C}$ will play an integral role in this discussion. We will note that since some of the proofs related to the general theory are too long to fit into an hour lecture, whenever a proof of some theorem is needed we will cite the location in [FH91] or [Bum04]. However despite the lack of proofs, just knowing the statements gives enough insight for one to carry out the representation theory of a semisimple Lie algebra in full generality. This lecture will attempt to combine Lectures 12 and 14 of [FH91].

Step 0: Verify your Lie algebra is semisimple

All of the following discussion will be useless if we don't first show our Lie algebra is semisimple. There are many equivalent conditions we can show:

Theorem. *The following are equivalent:*

(i) $\mathfrak{g}$ *is semisimple.*

(ii) the Killing form $\kappa(X,Y) = Tr(ad(X) \circ ad(Y))$ *is non-degenerate.*

(iii) $\mathfrak{g}$ *has no non-zero abelian ideals.*

(iv) $\mathfrak{g}$ *has no non-zero solvable ideals.*

(v) $Rad(\mathfrak{g})$ *is zero.*

The Killing form is a very powerful tool which we will not have time to discuss. We remark, however, that one useful property of the Killing form is its invariance under the adjoint representation:

$$B([X,Y],Z) + B(Y,[X,Z]) = 0.$$

We can compute the Killing form on $\mathfrak{sl}_n\mathbb{C}$ to be

$$\kappa(X,Y) = 2n\,\mathrm{Tr}(XY).$$

It is an easy exercise to verify this is non-degenerate, and thus, $\mathfrak{sl}_n\mathbb{C}$ is semisimple.

Step 1: Find the Cartan subalgebra

We recall for $\mathfrak{sl}_2\mathbb{C}$ we began by choosing a basis $\{H, L, R\}$ such that H was diagonal and decomposed an arbitrary representation into eigenspaces for H. For an arbitrary semisimple Lie algebra $\mathfrak{g}$ we would like to do the same thing, however one element will no longer suffice to obtain the whole picture. Instead we want to look at subalgebras $\mathfrak{h} \subset \mathfrak{g}$ where every element is simultaneously diagonalizable. In particular, recalling that commuting matrices are simultaneously diagonalizable, we want to look at abelian subalgebras $\mathfrak{h} \subset \mathfrak{g}$. We are led to our first definition.

Definition (The Cartan subalgebra)**.** A *Cartan subalgebra* of a (complex) semisimple Lie algebra $\mathfrak{g}$ is an abelian subalgebra all of whose elements are semisimple and is maximal with respect to these properties.

A key fact about Cartan subalgebras is that they exist and are unique up to conjugation (see [FH91] Appendix D). For $\mathfrak{sl}_3\mathbb{C}$ a Cartan subalgebra is simply given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} : a_1 + a_2 + a_3 = 0 \right\}. \tag{4.1}$$

Notice that $\mathfrak{h}$ is a 2-dimensional complex vector space. We define the rank of a semisimple Lie algebra $\mathfrak{g}$ to be the dimension of any of its Cartan subalgebras. In this case $\mathrm{Rank}(\mathfrak{sl}_3\mathbb{C}) = 2$ (we note this coincides with any other definition of rank of a Lie algebra).

Step 2: Decompose your Lie algebra

Our next step will be to use the Cartan subalgebra to decompose our Lie algebra. We once again draw a parallel to $\mathfrak{sl}_2\mathbb{C}$. Using the basis $\{H, L, R\}$ described in Lecture 3 we saw that a representation V of $\mathfrak{sl}_2\mathbb{C}$ decomposed into eigenspaces V_λ for the action of H, the action of L sent V_λ to $V_{\lambda-2}$, and the action of R sent V_λ to $V_{\lambda+2}$. Our first goal is to generalize this idea for a general semisimple Lie algebra $\mathfrak{g}$. We start by choosing some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Now let (π, V) be a representation of V. An eigenvector for $\mathfrak{h}$ will intutively mean some vector v that is an eigenvector for all $H \in \mathfrak{h}$. Concretely this means that there exists some $\alpha \in \mathfrak{h}^*$ such that

$$H(v) = \alpha(H) \cdot v. \tag{4.2}$$

for all $H \in \mathfrak{h}$. We thus define an eigenvalue for the action $\mathfrak{h}$ to be a linear functional $\alpha \in \mathfrak{h}^*$ such that there exists some nonzero vector $v \in V$ satisfying (4.2). We define the eigenspace associated to the eigenvalue α to be the subspace of all vectors $v \in V$ satisfying (4.2). Our first fact is the following:

Fact. *Any finite dimensional representation V of a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ admits a decomposition*

$$V = \bigoplus V_\alpha,$$

where V_α is an eigenspace for $\mathfrak{h}$ and α ranges over a finite subset of $\mathfrak{h}^*$.

The eigenvalues $\alpha \in \mathfrak{h}^*$ are known as the weights of the representation, the V_α are known as weight spaces, and the dimension of V_α is called the multiplicity of α in V.

In particular, consider the adjoint representation $(\mathrm{ad}, \mathfrak{g})$ of the Lie algebra. The above discussion gives a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_\alpha \right),$$

that is for any $H \in \mathfrak{h}$ and $X \in \mathfrak{g}_\alpha$ we have that $[H, X] = \alpha(H) \cdot X$. The *weights of the adjoint representation* are known as the *roots* of the Lie algebra. The corresponding $\mathfrak{g}_\alpha$ are known as the *root spaces*. We note that $\mathfrak{h}$ is the eigenspace corresponding to 0, that is, $\mathfrak{h} = \mathfrak{g}_0$. However $0 \in \mathfrak{h}^*$ will not usually be regarded as a root of the Lie algebra. The root spaces generalize the notion of L and R from $\mathfrak{sl}_2\mathbb{C}$ in the following sense:

if $X \in \mathfrak{g}_\alpha$, then the action of X sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$. This fundamental fact is a fairly easy computation which we carry out below, letting $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, and $H \in \mathfrak{h}$:

$$\begin{aligned} &[H,[X,Y]] + [X,[Y,H]] + [Y,[H,X]] = 0 && \text{(the Jacobi identity)} \\ \Longrightarrow &[H,[X,Y]] = [X,[H,Y]] + [[H,X],Y] && \text{(skew-symmetry)} \\ \Longrightarrow &[H,[X,Y]] = [X,\beta(H)\cdot Y] + [\alpha(H)\cdot X, Y] && (X \in \mathfrak{g}_\alpha \text{ and } Y \in \mathfrak{g}_\beta) \\ \Longrightarrow &[H,[X,Y]] = (\alpha(H)+\beta(H))\cdot[X,Y] && \text{(bilinearity of the Lie bracket).} \end{aligned}$$

Even more generally, given any representation V we can compute that the action of the root space $\mathfrak{g}_\beta$ sends V_α to $V_{\alpha+\beta}$. Let $X \in \mathfrak{g}_\beta$, $v \in V_\alpha$, and $H \in \mathfrak{h}$. We compute

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H,X](v) \\ &= X(\beta(H)\cdot v) + (\alpha(H)\cdot X)(v) \\ &= (\alpha(H)+\beta(H))\cdot X(v). \end{aligned}$$

We finally mention some key facts about root and root spaces without proof:

Facts.

(i) each root space $\mathfrak{g}_\alpha$ will be one dimensional.

(ii) the set of roots R will generate a lattice $\Lambda_R \subset \mathfrak{h}$ of rank equal to the dimension of $\mathfrak{h}$.

(iii) R is symmetric about the origin, that is, if $\alpha \in R$ then $-\alpha \in R$.

We remark that all the weights of an irreducible representation V must be congruent modulo the root lattice Λ_R. Otherwise for any weight α of V the space

$$V' = \bigoplus_{\beta \in \Lambda_R} V_{\alpha+\beta}$$

would be a proper subrepresentation of V.

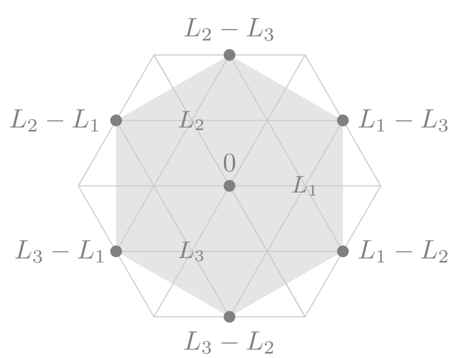
We now return to $\mathfrak{sl}_3\mathbb{C}$. We choose the Cartan subalgebra described by (4.1). With this Cartan subalgebra we can write

$$\mathfrak{h}^* = \frac{\mathbb{C}[L_1, L_2, L_3]}{(L_1+L_2+L_3)}$$

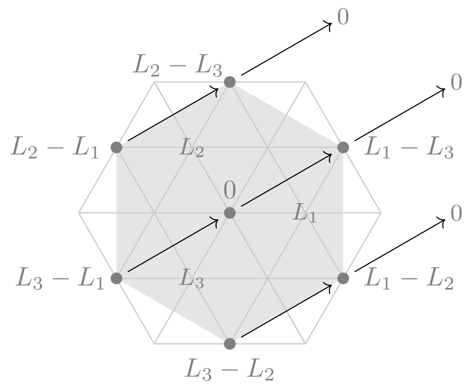
where

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i.$$

Now let $M \in \mathfrak{sl}_3\mathbb{C}$ and $H \in \mathfrak{h}$ arbitrary. We see that if M has entries m_{ij} then the commutator $[H,M]$ has entries $(a_i - a_j)m_{ij}$. We see that $[H,M]$ is a scalar multiple of M if and only if M has a single non-zero entry. Thus, if we let E_{ij} denote the 3×3 matrix with $(i,j)^{th}$ entry 1 and all other entries 0 we see that the E_{ij} generate the eigenspaces for $\mathfrak{h}$. In particular, the six matrices E_{ij} for $i \neq j$ correspond to the six roots $L_i - L_j$ and generate the root spaces $\mathfrak{g}_{L_i - L_j}$. We depict this below:



We remark from are discussion and the above diagram it is clear that the facts mentioned before, without proof, are satisfied. We also pictorially show the action of $\mathfrak{g}_{L_1-L_3}$ on the root spaces:



Pictures of the root diagrams essentially allow us to extract all the information of the Lie algebra.

Step 3: Find distinguished subalgebras isomorphic to $\mathfrak{sl}_2\mathbb{C}$

The analysis of the representation theory of $\mathfrak{sl}_2\mathbb{C}$ is fundamental to the representation theory of a general semisimple Lie group, which we now discuss. Let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be a root space. Then by previously discussed facts we have that $\mathfrak{g}_\alpha$ is one-dimensional, $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}$ is also a root space, and $\mathfrak{g}_{-\alpha}$ is one-dimensional. The commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is then a subspace of $\mathfrak{g}_0 = \mathfrak{h}$ of dimension at most one, and thus, its adjoint action carries each of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ to itself. It follows the direct sum

$$\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

is a subalgebra of $\mathfrak{g}$. We claim this subalgebra is isomorphic to $\mathfrak{sl}_2\mathbb{C}$, following from the two facts below:

Facts.

(i) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$; *and*

(ii) $[[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0$.

We now pick a basis $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ satisfying the commutator relations

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H, Y_\alpha] = -2Y_\alpha, \quad [X_\alpha, Y_\alpha] = H_\alpha.$$

We note that while X_α and Y_α are not uniquely determined, H_α is uniquely determined by the criterion $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\alpha(H_\alpha) = 2$.

Returning to $\mathfrak{sl}_3\mathbb{C}$, we find that the distinguished subalgebras $\mathfrak{s}_{L_i-L_j}$ are given by choosing $E_{ij} \in \mathfrak{g}_{L_i-L_j}$, $E_{ji} \in \mathfrak{g}_{L_j-L_i}$, and setting $H_{ij} = [E_{ij}, E_{ji}]$. It is simple computation to show that

$$[H_{ij}, E_{ij}] = 2E_{ij} \text{ and } [H_{ij}, E_{ji}] = -2E_{ji}.$$

We note it is clear from this example that $\mathfrak{s}_{L_i-L_j}$ and $\mathfrak{s}_{L_j-L_i}$ carry essentially the same information. Later we will create a clear distinction between the roots $\alpha \in R$ and $-\alpha \in R$ that will be useful for our analysis.

Step 4: Use the integrality of eigenvalues of the H_α

The first piece of information the subalgebras $\mathfrak{sl}_2\mathbb{C} \cong \mathfrak{s}_\alpha \subset \mathfrak{g}$ tells us that if we take any representation of V of $\mathfrak{g}$ and restrict to $\mathfrak{s}_\alpha$ the eigenvalues $\beta \in \mathfrak{h}^*$ must be integral with respect to H_α. From this we can conclude that every eigenvalue $\beta \in \mathfrak{h}^*$ of a representation of $\mathfrak{g}$ must assume integer values on all the H_α. We note that the set Λ_W consisting of all linear functionals $\beta \in \mathfrak{h}^*$ that are integer valued on all the H_α is a lattice, known as the *weight lattice*, with the property *all weights of all representations of* $\mathfrak{g}$ *will lie in* Λ_W. In particular $R \subset \Lambda_W$, and thus, $\Lambda_R \subset \Lambda_W$. One can show that in general, the root lattice will be a sublattice of finite index in the weight lattice. For $\mathfrak{sl}_3\mathbb{C}$ the weight lattice is generated by L_1, L_2, and L_3.

Step 5: Use the symmetry of the eigenvalues of the H_α

The second piece of information the subalgebras $\mathfrak{sl}_2\mathbb{C} \cong \mathfrak{s}_\alpha \subset \mathfrak{g}$ gives us is a little bit harder to deduce. In particular, recall that for $\mathfrak{sl}_2\mathbb{C}$ the eigenvectors for H were symmetric about the origin in $\mathbb{Z}$. This fact carries over to representations of the Lie algebra $\mathfrak{g}$. More concretely consider the involution on the vector space $\mathfrak{h}^*$ with +1-eigenspace the hyperplane

$$\Omega_\alpha = \{\beta \in \mathfrak{h}^* : \beta(H_\alpha) = 0\}$$

and -1-eigenspace the line spanned by α. Explicitly, W_α is the reflection in the plane Ω_α with axis the line spanned by α:

$$W_\alpha(\beta) = \beta - \frac{2\beta(H_\alpha)}{\alpha(H_\alpha)}\alpha = \beta - \beta(H_\alpha)\alpha.$$

Let $\mathcal{W}$ be the group generated by these involutions. We call $\mathcal{W}$ the Weyl group of the Lie algebra $\mathfrak{g}$.

We now make the claim that the *the set of weights of any representation of $\mathfrak{g}$ is invariant under the Weyl group*. In particular, suppose that V is an representation of $\mathfrak{g}$ with eigenspace decomposition $V = \bigoplus V_\beta$. We consider the equivalence classes of weights modulo α, that is $\beta \sim \beta'$ if and only if $\beta - \beta' = n\alpha$. The direct sum

$$V_{[\beta]} = \bigoplus_{n\in\mathbb{Z}} V_{\beta+n\alpha}$$

of the eigenspaces in an equivalence class will be a subrepresentation of V for $\mathfrak{s}_\alpha$. In particular, the set of weights of V congruent to any given β mod α will be invariant under W_α which is what we wanted to show.

Explicitely, consider the string of eigenvalues

$$\beta, \beta+\alpha, \beta+2\alpha, \ldots, \beta+m\alpha$$

corresponding to the nonzero weightspaces in the decomposition of $V_{[\beta]}$. Taking the action by H_α on this string we obtain

$$\beta(H_\alpha), \beta(H_\alpha)+2, \beta(H_\alpha)+4, \ldots, \beta(H_\alpha)+2m.$$

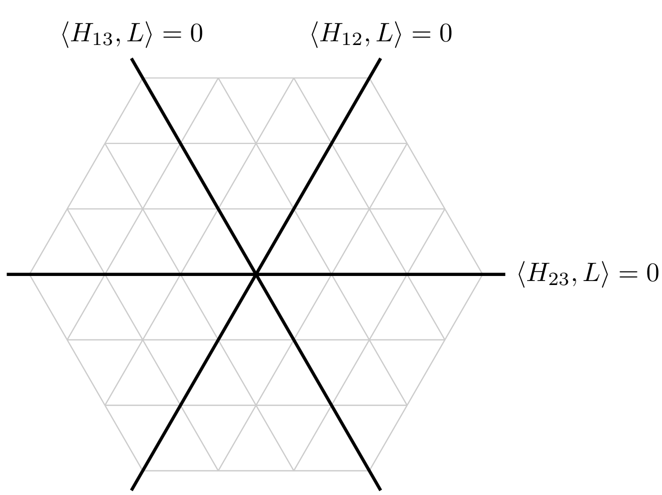
Since this must be symmetric about the origin, it follows $m = -\beta(H_\alpha)$. In particular

$$W_\alpha(\beta+k\alpha) = \beta + (-\beta(H_a) - k)\alpha = \beta + (m-k)\alpha.$$

The same analysis gives the multiplicity of the weights are invariant under the Weyl group. We conclude this discussion by mentioning one fact about the Weyl group:

Fact. *Every element of the Weyl group is induced by an automorphism of the Lie algebra $\mathfrak{g}$ carrying $\mathfrak{h}$ to itself.*

Returning to $\mathfrak{sl}_3\mathbb{C}$, our Weyl group $\mathcal{W}$ is generated by reflections about the bolded lines depected below:



Step 6: Choose a direction in $\mathfrak{h}^*$

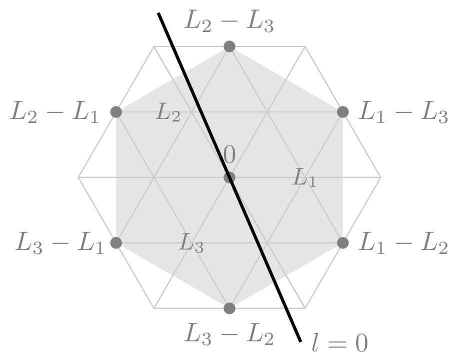
Recall that for each root $\alpha \in R$ the negative root $-\alpha \in R$. Our goal now is to decompose the set of roots into two sets

$$R = R^+ \cup R^-,$$

where intuitively R^+ captures the positive roots and R^- captures the negative roots. To do this we need to specify a direction, or orientation, of $\mathfrak{h}^*$. This is simply the choice of a real linear functional $l : \Lambda_R \to \mathbb{R}$ irrational with respect to the lattice Λ_R and extend by linearity to a functional $l : \mathfrak{h}^* \to \mathbb{R}$. We then declare $R^+ = \{\alpha \in R : l(\alpha) > 0\}$ to be the set of positive roots and $R^+ = \{\alpha \in R : l(\alpha) < 0\}$ to be the negative roots. The decomposition $R = R^+ \cup R^-$ is known as an *ordering of the roots*. For $\mathfrak{sl}_3\mathbb{C}$ consider the linear functional defined by

$$l(a_1L_1 + a_2L_2 + a_3L_3) = aa_1 + ba_2 + ca_3,$$

where $a + b + c = 0$ and $a > b > c$. The line $l = 0$ for such a transformation is shown below:



Under such a functional it follows $L_1 - L_3$, $L_2 - L_3$, and $L_1 - L_2$ are the positive roots and $L_3 - L_1$, $L_3 - L_2$, and $L_2 - L_1$ are the negative roots.

The decomposition allows us to once again define what we mean by highest weight vector:

Definition (Highest Weight Vector)**.** Let V be any representation of $\mathfrak{g}$. A nonzero vector $v \in V$ that is both an eigenvector for the action of the Cartan subalgebra $\mathfrak{h}$ and in the kernel of the action of $\mathfrak{g}_\alpha$ for all $\alpha \in R^+$ is called a *highest weight vector*.

We then have the following theorem:

Theorem. *For any semisimple complex Lie algebra* $\mathfrak{g}$*,*

(i) every finite-dimensional representation V of $\mathfrak{g}$ possesses a highest weight vector;

(ii) the subspace W of V generated by the images of a highest weight vector v under successive applications of the action by root spaces $\mathfrak{g}_\beta$ for $\beta \in R^-$ is an irreducible subrepresentation.

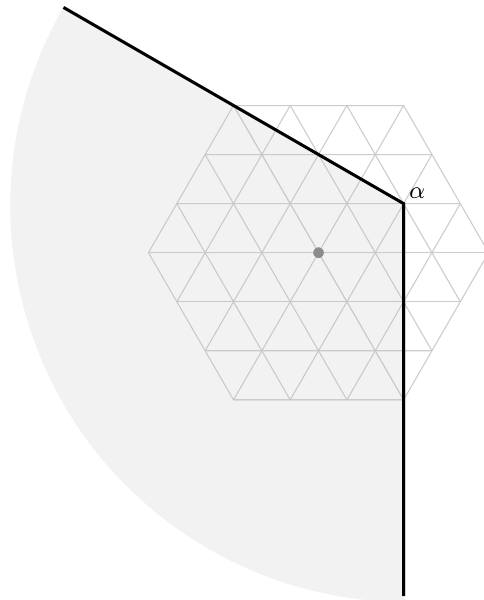
(iii) an irreducible representation possesses a unique highest weight vector up to scalars

Proof. Part (i) is easy. Choose α to be the weight appearing in the decomposition of V such that $l(\alpha)$ is maximal and choose any nonzero vector $v \in V_\alpha$. For any root $\beta \in R^+$ we have $V_{\alpha+\beta} = \{0\}$ by maximality of α, and thus, v is in the kernel of the action of $\mathfrak{g}_\beta$ as desired.

Part (ii) follows from induction. Let W_n be the subspace spanned by all $w_n \cdot v$ where w_n is a word of length at most n in elements of $\mathfrak{g}_\beta$ for negative β. We then want to show for any X in any positive root space $X \cdot W_n \subset W_n$. To see this write a generator of W_n in the form $Y \cdot w$, where Y is in some negative root space and $w \in W_{n-1}$. Then use the commutator relation $X \cdot Y \cdot w = Y \cdot X \cdot w + [X, Y] \cdot w$. The claim now follows by induction.In particular, by the inductive hypothesis we have $X \cdot W_{n-1} \subset W_{n-1}$ so $Y \cdot X \cdot w \in W_n$. Similarly, assuming that $X \in \mathfrak{g}_\beta$ and $Y \in \mathfrak{g}_\gamma$ then $[X, Y] \in \mathfrak{g}_{\beta+\gamma}$. If $l(\beta + \gamma) \geq 0$ then $[X, Y] \cdot w \in W_{n-1}$. If not then $[X, Y] \cdot w \in W_n$. It follows in either case that $X \cdot Y \cdot w \in W_n$. Now let $W \subset V$ be the union of all the W_n's. Then W is a subrepresentation, and if $W = W' \oplus W''$, then either W' or W'' will have to contain the one-dimensional weight space W_α, and thus, will equal all of W.

Part (iii) is immediate. If $v \in V_\alpha$ and $v \in V_\beta$ are two highest weight vectors, not scalar multiples of each-other, we would have $l(\alpha) > l(\beta)$ and vice-versa. This is a contradiction. ■

For $\mathfrak{sl}_3\mathbb{C}$ the theorem implies that given some highest weight $\alpha \in \mathfrak{h}^*$ for a representation V, all other weights $\beta \in \mathfrak{h}^*$ occurring in V lie in essentially a $\frac{1}{3}$-plane with corner at α:

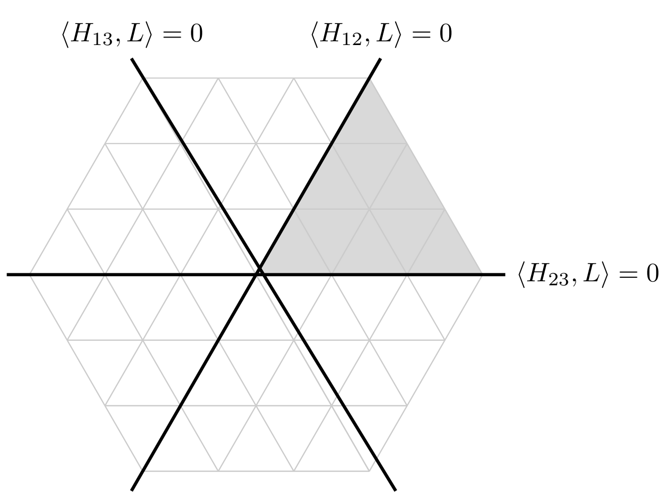


We call the weight of the highest weight vector for an irreducible representation the highest weight of that representation. There is also one more useful piece of terminology. We call a positive (respectively, negative) root $\alpha \in R$ *primitive* or *simple* if it cannot be expressed as a sum of two positive (respectively, negative) roots. It turns out we have the following:

Fact. *Any irreduicble representation V is generated by the images of its highest weight vector v under successive applications of root spaces $\mathfrak{g}_\beta$ where β rangesover the primitive negative roots.*

We also note that *every vertex of the convex hull of the weights of V must be conjugate to α under the Weyl group*. In particular, the set of weights of V will be exactly the weights that are congruent to α modulo the root lattice Λ_R and that lie in the convex hull of the images of α under the Weyl group.

We finish off with a final bit of terminology. By the previous discussion, the highest weight of any representation V will be a weight α satisfying $\alpha(H_\beta) \geq 0$ for all $\beta \in R^+$. The locus $\mathscr{W}$, in the real span of the roots, of points satisfying these inequalities is called the *Weyl chamber* associated to the ordering of the roots. For our chosen ordering of the roots of $\mathfrak{sl}_3\mathbb{C}$ the Weyl chamber is depicted below:



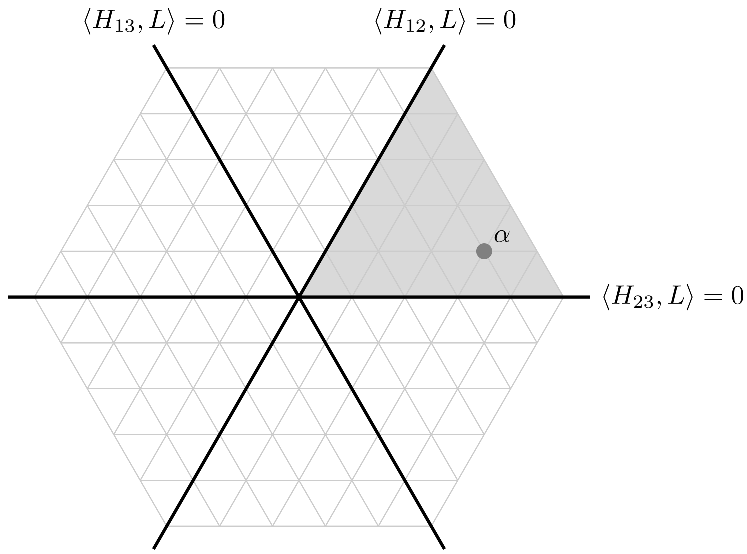
Note we could of also described a Weyl chamber as the closure of a connected component of the complement of the union of the hyperplanes Ω_α. A fact, which we wont prove, is that the Weyl group acts simply transitively on the set of Weyl chambers, and thus, on the set of orderings of the roots.

Step 7: Classify the irreducible, finite-dimensional representations

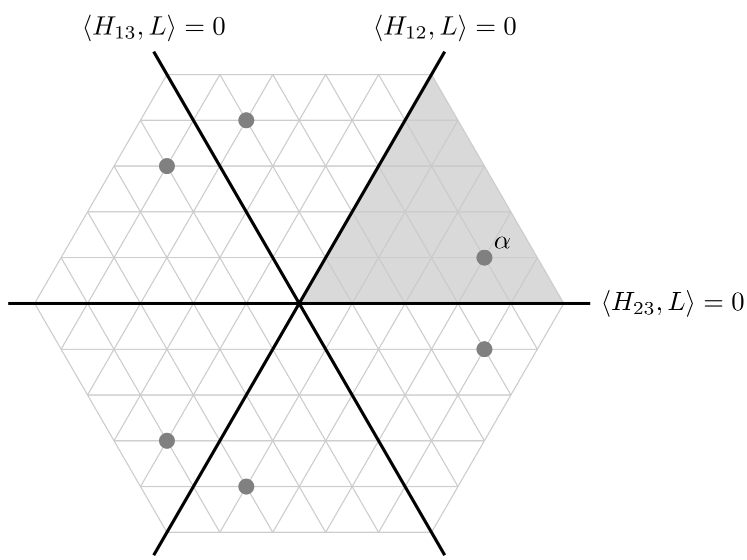
All of the preceding discussion now allows us to state (without proof) the fundamental existence and uniqueness theorem for irreducible finite-dimensional representations of a semisimple Lie algebra $\mathfrak{g}$. We have the following:

Theorem. *For any α in the intersection of the Weyl chamber $\mathscr{W}$ associated to the ordering of the roots with the weight lattice Λ_W, there exists a unique irreducible, finite-dimensional representation Γ_α of $\mathfrak{g}$ with highest weight α; this gives a bijection between $\mathscr{W} \cap \Lambda_W$ and the set of irreducible representations of $\mathfrak{g}$. The weights of Γ_α will consist of those elements of the weight lattice congruent to α modulo the root lattice Λ_R and lying in the convex hull of the set of points in $\mathfrak{h}^*$ conjugate to α under the Weyl group.*

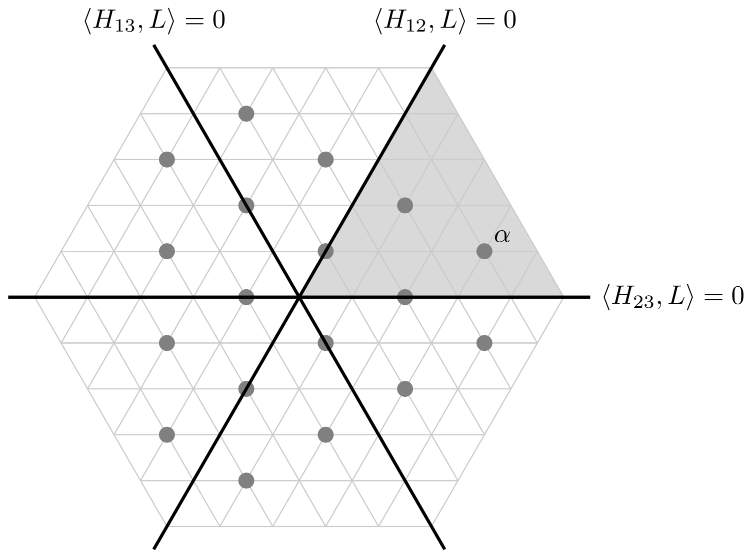
We unpack this theorem for $\mathfrak{sl}_3\mathbb{C}$. We begin by choosing some root $\alpha \in \mathscr{W} \cap \Lambda_W$:



We now allow the Weyl group to act on this weights:



Finally we fill in the convex hull of this set of weights:



The diagram above corresponds to the weights of the irreducible representation Γ_α.

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