Lecture 2

Representations and the Baker-Campbell-Hausdorff Formula

Representation Theory

Representation theory is a very powerful tool in mathematics, allowing for difficult questions that arise in studying abstract algebra to be transferred to the much more understood subject of Linear algebra. In fact, representation theory finds many uses not only in abstract algebra, but also analysis, geometry, number theory, and physics. Now that we have defined a Lie group and a Lie algebra we are ready to dive into the meat of the lecture and discuss their corresponding representation theory. We begin with general definitions pertaining to group representations.

> **Definition** (Group Representation)**.** Let K be a field. An K-representation of a group G is a pair (π, V) where V is a vector space over K and $\pi : G \to \mathrm{GL}(V)$ is a homomorphism of groups. The dimension of a representation is equal to the dimension of V as a vector space.

Note that a representation of G defines a G-module structure on V. Thus, when the underlying representation (π, V) is understood, we will use the shorthand $\pi(g)v = gv$ for $g \in G$ and $v \in V$. Since a representation of G gives V a G-module structure we can define a homomorphism between two representations to be a G-linear map on V in the following sense:

> **Definition** (Morphism of Group Representations)**.** Let G be a group and (π_1, V), (π_2, W) be two K-representations of G. A homomorphism between (π_1, V) and (π_2, W) is a linear transformation $T : V \to W$ such that $T(\pi_1(g)v) = \pi_2(g)T(v)$ for all $g \in G$ and $v \in V$.

Maps as described above are often called G-equivariant maps or intertwining maps. Two K-representations of a group G are said to be isomorphic if there exists an invertible intertwining map between them.

We now discuss the above in the context of Lie groups. First, since G admits a smooth structure we are mainly interested in fields K such that for a vector space V over K the group $\mathrm{GL}(V)$ also admits a smooth structure. Thus when speaking about Lie groups the natural choices for K are the field of real numbers $\mathbb{R}$ and the field of complex numbers $\mathbb{C}$. Second, given $\mathrm{GL}(V)$ is a Lie group we require our representations (π, V) be smooth (i.e. π is a Lie group homomorphism). Given these two conditions we can differentiate π to obtain a Lie algebra map $d\pi_e : \mathfrak{g} \to \mathfrak{gl}(V)$. This leads us to define the following:

> **Definition** (Lie Algebra Representation)**.** A representation of a real (complex) Lie algebra $\mathfrak{g}$ is a pair (α, V) where V is a real (complex)vector space and $\alpha : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra map. The dimension of a representation is equal to the dimension of V as a vector space. (Recall $\mathfrak{gl}(V) = \mathrm{End}(V)$ with the commutator bracket).

Intertwining maps for Lie algebra representations are defined analogously to that of groups.

Definition (Morphism of Lie algebra Representations)**.** Let $\mathfrak{g}$ be a Lie algebra and (α_1, V), (α_2, W) be two representations of $\mathfrak{g}$. A homomorphism between (α_1, V) and (α_2, W) is a linear transformation $T : V \to W$ such that $T(\alpha_1(X)v) = \alpha_2(X)T(v)$ for all $X \in \mathfrak{g}$ and $v \in V$.

Also analogously to groups, we say two representations of $\mathfrak{g}$ are isomorphic if there exists an invertible intertwining map between them.

We state a couple facts about the above discussion. First, the Jacobi Identity is equivalent for Lie algebras arising from Lie groups to the statement that $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation. We will thus often refer to this map as the adjoint representation of a Lie algebra. Second, a representation is called faithful if it is injective. We state, but not prove, the following fundemental result about faithful representations of Lie algebras:

Theorem (Ado's Theorem)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the real (or complex) numbers. Then there exists some finite dimensional real (or complex) vector space V such that $\mathfrak{g}$ has a faithful representation into V.*

Ado's theorem lets us regard any finite dimensional Lie algebra over the real (or complex) numbers as some set of square matrices over a vector space with bracket the commutator. We will use this fact repeatedly throughout this lecture.

The last thing we discuss is invariant subspace of a representation. Let (π, V) be a representation of a group G. We say that subspace $W \subset V$ is invariant with respect to (π, V) if $gW \subset W$ for all $g \in G$. If (π, V) has no proper invariant subspaces we call the representation irreducible. We remark that the same definition of invariant subspace carries over to representations of a Lie algebra $\mathfrak{g}$, and in fact, if $W \subset V$ is invariant under a representation (π, V) of G then it is also invariant under the representation $(d\pi_e, V)$ of $\mathfrak{g}$. Thus, a representation (π, V) is irreducible if and only if $(d\pi_e, V)$ is irreducible.

Isogeny

We depart for a moment from the previous discussion to talk about some general topology we will need to continue our study. Let X be a topological space. We will first define what we mean when we say X is a simply connected space.

Definition (Simply Connected)**.** A space X is said to be *simply connected* if and only if it is path-connected and any loop in X defined by $f : S^1 \to X$ can be contracted to a point, that is, there is a map $F : D^2 \to X$ such that $F|_{\partial D^2} = f$. Equivalently, X is path-connected and trivial fundamental group at each point.

We also define what we mean by a covering space of X.

Definition (Covering Space)**.** A covering space of a topological space X is a topological space $\widetilde{X}$ together with a map $p : \widetilde{X} \to X$ satisfying the following condition: each point $x \in X$ has an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open subsets in $\widetilde{X}$ each of which is mapped homeomorphically onto U by p. In the case $\widetilde{X}$ is simply connected it is called the universal cover of X and is unique up to homeomorphism respecting the covering space maps.

Let $p : \widetilde{X} \to X$ be a covering space and $f : Y \to X$. We say that $\widetilde{f} : Y \to \widetilde{X}$ is a lift of f if $\widetilde{f} \circ p = f$. It turns out we can completely characterize when a map $f : Y \to X$ will lift to a map $\widetilde{f} : Y \to \widetilde{X}$, see [Hat02] for details. This lifting property lets us deduce the following proposition:

Proposition. *Let G be a Lie group, H a connected manifold, and $p : H \to G$ a covering space map. Let e' be a lift of the identity e in G (i.e. $p(e') = e$). Then there is a unique Lie group structure on H such that e' is the identity and p is a map of Lie groups; and the kernel of p is in the center of H.*

The above is deduced by lifting the multiplication map $G \times G \to G$ to the universal cover $\widetilde{G}$ and using the following "converse" proposition for the intermediate covers:

Proposition. *Let H be a Lie group, and $\Gamma \subset Z(H)$ a discrete subgroup of its center. Then there is a unique Lie group structure on the quotient group $G = H/\Gamma$ such that the quotient map $H \to G$ is a Lie group map.*

We now define an equivalence relation on Lie groups. We call a Lie group map $G \to H$ an isogeny if it is a covering space map and call G and H isogenous if there is an isogeny between them. Isogeny is not quite an equivalence relation, however we take the equivalence relation it generates. We note two key features of the isogeny equivalence relation. First, every isogeny equivalence class of a Lie group G has an initial member given by the universal cover $\widetilde{G}$. Second, two isogenous Lie groups have isomorphic Lie algebras. We will see later that every finite dimensional Lie algebra can be associated to a unique isogeny class of Lie groups.

Returning to the Exponential Map

Last time we constructed for any arbitrary Lie group G and its Lie algebra $\mathfrak{g}$ the unique homomorphisms $\rho_X : \mathbb{R} \to G$ (one-parameter subgroups) satisfying $\rho_X(0) = e$ and $\rho'_X(0) = X$ for each $X \in \mathfrak{g}$ and defined $\rho_X(t) := e^{tX}$ and $\exp : \mathfrak{g} \to G$ as the map $\exp(X) = e^X$. The map $\exp$ is called the exponential of the Lie algebra. We note in general $\exp$ is not surjective, however given our Lie group is connected this does not matter as a result of the following theorem:

Theorem. *The exponential map is the unique map from $\mathfrak{g}$ to G taking 0 to e whose differential at the origin $d\exp_0 : T_0\mathfrak{g} = \mathfrak{g} \to T_eG = \mathfrak{g}$ is the identity, and whose restrictions to the lines through the origin in $\mathfrak{g}$ are one-parameter subgroups of G.*

The theorem follows essentially from the fact that any line through the origin of $\mathfrak{g}$ can be written as tX for some $X \in \mathfrak{g}$ and the fact $\rho_{tX}(s) = \rho_X(ts)$ by uniqueness of one-parameter subgroups. In particular, we can deduce from this that given a homomorphism $\Psi : G \to H$ the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d\psi_e} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\psi]{} & H \end{array}$$

commutes. Since $d\exp_0$ is an isomorphism, it follows from the inverse function theorem that $\exp$ is a local diffeomorphism at the origin of $\mathfrak{g}$ and thus maps a sufficiently small open disc $\Delta \subset \mathfrak{g}$ centered at the origin of $\mathfrak{g}$ diffeomorphically onto a open set $\exp(\Delta) \subset G$. If G is connected then $\exp(\Delta)$ generates all of G by the theorem proven last time, and thus according to the above diagram, ψ is completely determined by $d\psi_e$.

Now let $X \in \mathfrak{gl}(V)$ where V is a real (or complex) vector space. Recall from elementary calculus that for real numbers we can represent the exponential e^t by its taylor series, i.e.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!},$$

for all values of t. We now define $X^n = \prod_{i=1}^n X$ and define

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

It turns out this map converges for all $X \in \mathfrak{gl}_n(V)$ and is a map into $\mathrm{GL}(V)$. The map defined above is referred to as the matrix exponential. It is simple computation to show that $e^{sX}e^{tX} = e^{(s+t)X}$, $e^0 = I$, and $\left.\frac{d}{dt}e^{tX}\right|_{t=0} = X$. We can therefore conclude that e^{tX} is precisely the one-parameter subgroup corresponding to X. Therefore we deduce if $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$, the exponential map is precisely the matrix exponential. We will use this fact in the next section to show that the group law on G is locally determined by the Lie algebra structure on $\mathfrak{g}$.

The Baker-Campbell-Hausdorff Formula

We now know that a Lie group map is completely determined by its differential at the identity, but now we would like to classify which Lie algebra maps arise as differentials of a Lie group map, the end goal being the construction of a Lie group representation from a Lie algebra representation. We note currently that we currently only have one tool linking a Lie group G to its Lie algebra $\mathfrak{g}$, the exponential map $\exp : \mathfrak{g} \to G$. Our first goal then should be an attempt to recover the group operation of G simply from the Lie algebra structure of $\mathfrak{g}$. That is, we want to define some partial function (maybe total) $* : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$e^{X*Y} = e^X e^Y.$$

To begin, we will use Ado's Theorem to view $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{gl}(V)$. We can then view the exponential map as the matrix exponential so that

$$\begin{aligned} e^{X*Y} &= e^X e^Y \\ &= \left(I + X + \frac{X^2}{2} + \cdots\right)\left(I + Y + \frac{Y^2}{2} + \cdots\right) \\ &= I + (X+Y) + \left(\frac{X^2}{2} + XY + \frac{Y^2}{2}\right) + \cdots \end{aligned}$$

If $G \subset \mathrm{GL}(V)$ we can define for $g \in G$ sufficiently close to the identity $I \in G$ a partial inverse to the exponential map given by

$$\log(g) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(g-I)^n}{n}.$$

We can use this to compute

$$X * Y = \log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\big([X,[X,Y]] + [Y,[Y,X]]\big) - \cdots.$$

This formula is known as the Baker-Campbell-Hausdorff formula and can be completely expressed in terms of the Lie bracket. The Baker-Campbell-Hausdorff formula may not converge for all $X, Y \in \mathfrak{g}$ but will for X and Y suitably small (i.e. small enough that $\log(e^X e^Y)$ is defined).

While we have not written out a closed form for the Baker-Campbell-Hausdorff formula, the existence of the formula is more important than actually using it in computation. The first theorem we can prove with this formula is the following:

Theorem. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then the subgroup of the group G generated by $\exp \mathfrak{h}$ is an immersed subgroup H with tangent space $T_e H = \mathfrak{h}$.*

The basic idea is that we need only look at the exponential of small open discs $\Delta \subset \mathfrak{h}$ about the origin and prove that $\exp(\Delta) \cdot \exp(\Delta) \subset \exp \mathfrak{h}$ (since $\exp(\Delta)$ and $\exp \mathfrak{h}$ will generate the same subgroup). We now use the fact that G is isogenous to some subgroup of $\mathrm{GL}(V)$ (via Ado's Theorem) and the Baker-Campbell-Hausdorff formula to deduce the theorem.

The next two theorems essentially will prove that the category of Lie algebras is equivalent to the full subcategory of Lie groups consisting only of simply connected Lie groups. The first result is known as Lie's second theorem:

Theorem (Lie's Second Theorem)**.** *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, such that G is simply connected. If $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra map there is a unique map $F : G \to H$ of Lie groups such that $dF_e = f$.*

Proof. The idea of this proof is to consider the product $G \times H$ with Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$ and notice that f being a map of Lie algebras is equivalent to the graph $\mathfrak{j} \subset \mathfrak{g} \oplus \mathfrak{h}$ of f being a Lie subalgebra. Using the previous theorem we associate an immersed subgroup $J \subset G \times H$ to $\mathfrak{j}$. Now projection $\pi : J \to G$ induces an isomorphism between $\mathfrak{j}$ and $\mathfrak{g}$, and thus, π is an isogeny (via Lie's Third Theorem to be stated). Since G is simply connected π is an isomorphism so $J \cong G$. Finally, the projection $\eta : G \to H$ onto the second factor is the desired map. ■

The next result is known as Lie's Third theorem:

Theorem (Lie's Third Theorem)**.** *If $\mathfrak{g}$ is a finite dimensional Lie algebra there exists a unique simply connected Lie group G with Lie algebra $\mathfrak{g}$.*

Proof. Use Ado's Theorem to embed $\mathfrak{g}$ into $\mathfrak{gl}(V)$. Then there exists an immersed Lie subgroup $G_0 \subset \mathrm{GL}(V)$ with Lie algebra $\mathfrak{g}$ (by a previous theorem). Take the univeral cover G of G_0. G is simply connected and has Lie algebra $\mathfrak{g}$. Uniqueness follows from the fact that the isomorphism class of a simply connected Lie group is determined by its Lie algebra. ■

Using the above two theorems we can establish the following principle:

Representations of a simply connected Lie groups are in bijective correspondence with representations of finite dimensional Lie algebras.

In order to study the representation theory of Lie groups we therefore need only to study the representation theory of Lie algebras.

Representations of Lie Algebras

With the justification out of the way we are now ready to study the representation theory of Lie algebras. We begin with a long list of definitions that will come in handy later.

Definitions. Let $\mathfrak{g}$ be a Lie algebra. We make the following defintions:

(I) the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is the subspace of $\mathfrak{g}$ consisting of elements $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$.

(II) $\mathfrak{g}$ is abelian if $Z(\mathfrak{g}) = \mathfrak{g}$, i.e. if all Lie brackets are zero.

(III) a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

(IV) the lower central series $\mathcal{D}_k\mathfrak{g}$ is defined by $\mathcal{D}_1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathcal{D}_k = [\mathcal{D}_{k-1}\mathfrak{g}, \mathfrak{g}]$. (Note each $\mathcal{D}_k\mathfrak{g}$ is an ideal of $\mathfrak{g}$)

(V) the derived series $\mathcal{D}^k\mathfrak{g}$ is defined by $\mathcal{D}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathcal{D}^k = \left[\mathcal{D}^{k-1}\mathfrak{g}, \mathcal{D}^{k-1}\mathfrak{g}\right]$. (Note each $\mathcal{D}^k\mathfrak{g}$ is an ideal of $\mathfrak{g}$)

(VI) $\mathfrak{g}$ is nilpotent if $\mathcal{D}_k\mathfrak{g} = 0$ for some k.

(VII) $\mathfrak{g}$ is solvable if $\mathcal{D}^k\mathfrak{g} = 0$ for some k.

(VIII) $\mathfrak{g}$ is semisimple if $\mathfrak{g}$ has no nonzero solvable ideals.

(IX) $\mathfrak{g}$ is simple if $\dim(\mathfrak{g}) > 1$ and it contains no nontrivial ideals.

We commonly refer to $\mathcal{D}^1 = \mathcal{D}_1$ as the commutator subalgebra of $\mathfrak{g}$ and denote it via $\mathcal{D}\mathfrak{g}$. We also note that $\mathfrak{h} \subset \mathfrak{g}$ induces a Lie bracket on the quotient vector space $\mathfrak{g}/\mathfrak{h}$ if and only if $\mathfrak{h}$ is an ideal. One thing we could prove is that $\mathfrak{g}$ is solvable if and only if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable for ideals $\mathfrak{h} \subset \mathfrak{g}$. In particular, this allows us to deduce that the sum of any two solvable ideals in a Lie algebra are once again solvable. We define the radical of $\mathfrak{g}$, Rad($\mathfrak{g}$), to be the sum of all solvable ideals in $\mathfrak{g}$. Rad($\mathfrak{g}$) then becomes a maximal solvable ideal in $\mathfrak{g}$. The quotient $\mathfrak{g}$/Rad($\mathfrak{g}$) is semisimple, and we obtain an exact sequence

$$0 \to \text{Rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\text{Rad}(\mathfrak{g}) \to 0$$

With this as preliminary justification, in order to study the representation theory of an arbitrary Lie algebra $\mathfrak{g}$ we essentially need to study the representation theory of solvable and semisimple Lie algebras.

We begin with the case $\mathfrak{g}$ is solvable. The representation theory of $\mathfrak{g}$ starts with a proof of Engel's theorem:

Theorem (Engel's Theorem)**.** *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be any Lie subalgebra such that every $X \in \mathfrak{g}$ is a nilpotent endomorphism of V. Then there exists a nonzero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$.*

From here we deduce Lie's theorem:

Theorem (Lie's Theorem)**.** *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a complex solvable Lie algebra. Then there exists a nonzero vector $v \in V$ that is an eigenvector of X for all $X \in \mathfrak{g}$.*

Lie's theorem now lets us deduce the irreducible representations of a complex solvable Lie algebra are all one dimensional. In fact, we can deduce even more:

Theorem. *Let $\mathfrak{g}$ be a complex Lie algebra, $g_{ss} = \mathfrak{g}/Rad(\mathfrak{g})$. Every irreducible representation of $\mathfrak{g}$ is of the form $V = V_0 \otimes L$, where V_0 is an irreducible representation of $\mathfrak{g}_{ss}$ and L is a one-dimensional representation.*

We now restrict our attention for the moment to semisimple Lie algebras. A key fact we will make use of is:

Theorem (Complete Reducibility)**.** *Let V be a representation of the semisimple Lie algebra $\mathfrak{g}$ and $W \subset V$ a subspace invariant under the action of $\mathfrak{g}$. Then there exists a subspace $W' \subset V$ complementary to W and invariant under $\mathfrak{g}$.*

This allows us to decompose any representation of a semisimple Lie algebra into a direct sum of irreducible representations. Thus, in order to study the representation theory of a semisimple Lie algebra we need only find it's irreducible representations.